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REVIEW

A Review of Mixed Lubrication Modelling and Simulation

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A B S T R A C T

Majority of the rolling contacts applied in complex interacting machine elements for example bearings and gears perform under Mixed Lubricating (ML) conditions, where the lubricant film can't fully separate the asperities of the two contacting surfaces. Highly loaded, interacting asperity surrounded with lubricant film, leads to the development of surface originated defects such as scuffing, micropitting, and wear in the ML region. This region exists amid Elastohydrodynamic Lubrication and Boundary Lubrication which needs consolidated knowledge of fluid film and direct contact of asperities, this makes the problem more difficult to solve numerically. Numerous authors have used the Reynolds equation or its modified versions to solve the lubrication problem numerically. However, still, some uncertainty is there to model mixed lubrication operating conditions, with traditional Reynold's equation, because the assumptions commonly made in Elastohydrodynamic lubrication are not valid within the context of mixed lubrication regime. In this paper previously, used models for mixed lubrication have been examined, and various development in related fields are discussed. Therefore, this review will provide an integrated, synthesized overview of the topic and in turn will lead to benefits for wide-ranging academic, industrial and research communities.

1. INTRODUCTION

Machines are often subjected to extreme operational conditions such as high loading configuration, high temperature, vibrations, thin film. Movement of interacting surfaces generates friction forces, which are responsible for high heat generation, thereby causing an adverse effect on the surface. For example, heat generation due to friction results in sharp local temperature rise, adhesion and wear in rolling/sliding contact [1]. Modelling and Simulation of lubrication processes such as contact mechanism, friction and wear are such areas that require much improvement. Accurate prediction of these processes can improve the performance of interfaces. Physical conditions between the contact are very complex for example collision of surface asperities, mechanical deformation, lubricant flow, and chemical reactions. The Lubricant theory has been proposed to model and analyses the film thickness and pressure variations between the contact area [2].

This tribological area is widespread for example, rolling element bearings, gears, cam and followers, contact between human joints, etc.

Both Boundary Lubrication (BL) and Elastohydrodynamic Lubrication (EHL) are bridged together by Mixed lubrication (ML), which confirms local events such as solid to solid (asperity) contact occur under severe loading conditions [3]. Figure 1 shows different lubrication conditions During initial loading conditions (running-in), the area of actual contact between asperity tips is smaller than the apparent area due to micro-contacts. As a result, they produce high concentrated pressure, causing the asperities to deform plastically (Figure 1. (b)) until such time when enlarged bearing contact area is enabled to sustain aplied load [4]. Furthermore, these asperity contacts surrounded by lubricant films initiate surface pitting and scuffing failure on contacting surfaces. In addition, these surface-originating defects are the major cause of failures associated with Rolling Contact Fatigue (RCF)[5].

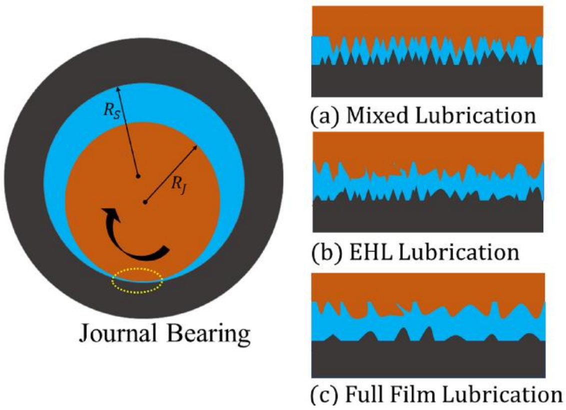

Fig. 1. Different lubrication conditions.

Over the years, there has been an increased interest in modelling and analysis of the contacts working under mixed lubrication, to avoid severe surface damage. Reynolds Equation, which is based on thin-film approximation, can be employed to define lubricant flow at the interface of the contacting surfaces [6]. It is a simplified form of momentum equation after applying many unrealistic assumptions. For example, a fluid film is always present between the surfaces to separate them, which is true for hydrodynamic lubrication (Figure 1. (c)). However, to model flow in mixed lubrication asperity contact must be considered. This means lubricant film is zero when asperities are making contact (Figure 1. (a)). Therefore, it's ambiguous that general Reynold's equation is capable to predict true pressure variation in mixed lubrication. In this paper, we discuss different models that have been proposed previously for numerical simulation of rolling/sliding contacts working under mixed lubrication.

Flow, pressure, stress distribution, friction, wear, and adhesion are significantly impacted by the roughness of interacting surfaces. There have been numerous researches to study the influence of roughness of lubricated surfaces [7-16]. Surface roughness refers to how far a surface is from an ideal or flawless geometrical surface, whether it is at the microscopic level or the macro level. Hydrodynamic roughness parameter ($\lambda = h_{cs}/\sigma$) known as lambda ratio (λ) determines the intensity of roughness on average film thickness in terms of influence. This parameter is obtained from minimum interfacial film thickness and compound surface roughness ratio between two interacting surfaces [17]. When λ values are relatively small, then the apparent effects of roughness are significant. However, the roughness effects become very limited or negligible when $\lambda > 3$. On the other hand no investigations and corresponding results have been reported in any published literature which takes an account for conditions where $\lambda < 0.5$, this mainly attributed to the intrinsic challenges when it comes to stochastic models which are not capable to deal with deep surface asperities' penetration in extreme conditions posed by heavy surface interaction [18]. In general, the λ ratio provides an estimate of the lubrication regime, but it has a lot of limitations [19]; however, it is still largely employed in engineering and research e.g. for predicting surface-initiated damage in bearings and in the design of tribological experiments. Various definitions of λ exist; it can be either minimum or central film thickness, vs. e.g., Ra or Rq (r.m.s) roughness parameter. Therefore, there is no clear and precise value of this parameter that defines the transition between the lubrication regimes. Several researchers have studied synergistic surface texturing and roughness impacts on frictional performance of tribo-pairs working under Mixed Lubrication and Hydrodynamic (HDL) regimes [20-24]. Researchers discovered that under mixed lubrication (ML), surface roughness can have a considerable impact on friction coefficient, but under hydrodynamic lubrication (HL), its influence can be ignored. [25]. Stribeck curve in figures 2 shows various lubrication regimes. Friction variation trend in Stribeck curve depends on the dimensionless number, known as Hersey number, which is a relation between viscosity (η), rotational speed (N) and average load (P). Stribeck curve can be generated by experimental results or numerical computations under wide range of operating conditions, that makes the procedure more challenging. Zhu et al. [26] produced a Stribeck curve using numerical data for rough counter formal contact. Further, evidence including experimental data shows that the Stribeck curve for counter formal contact interface behaves differently from those for conformal contacts [27].

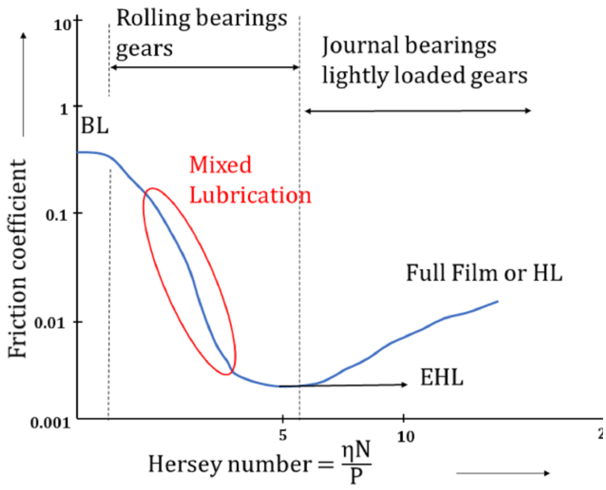

Fig. 2. Stribeck curve.

Three possibilities exist to study the roughness effect in lubrication [28].

1. First, when speed is high the full film of oil supports the load and generate a working clearance at the interface, thus roughness effects are very less in the Hydrodynamic Lubrication ($\lambda > 3$).
2. As a result of high pressure and increased viscosity, the rough surface's peak is partially deformed, but the external load is fully supported by the fluid film and is known as Elastohydrodynamic Lubrication (EHL).
3. At high load and low speeds asperities (roughness peaks) are difficult to completely flatten, which allow asperities interaction, and are surrounded by a lubricant film, which allows the external load to be distributed partially over the oil film and contacting asperities ($\lambda < 0.5$).

Researchers have already explored the first two areas numerically or experimentally and have found satisfactory results using the Reynolds equation in conjunction with changes in viscosity and temperature, which are not considered in this review paper. This paper focuses specifically on the third area, where fluid film coexists with asperity contacts, which in turn leads to enhanced wear and reduced service life of engineering products.

Hence this review starts with a succinct outline of Reynolds equation and its modified versions to depart from some of the assumptions used. The lubrication is affected by roughness, therefore there was a need to numerically model the surface roughness, which can help to understand the lubrication for the rough contacting surface. Many models have been developed in past for modelling roughness, which is based on some statistical parameters. The advantages and drawbacks of these models are outlined. In this paper, several new approaches have been explored, such as measurement of film thickness by optical interferometry, molecular dynamics, CFD, numerical simulation to investigate roughness effects on lubrication.

2. REYNOLD'S EQUATION

Reynold's equation is widely accepted to describe flow between two contacting surfaces. In essence, the equation is derived by simplifying the Navier-Stokes (NS) equation, which is based on the continuum hypothesis. As a result, NS equations are very complicated since they include inertia effects, body forces, viscous effects, and pressure terms. which make analytical solutions for practical problems difficult. Therefore, the equations have been simplified by using non-Dimensionalisation, thereby reducing the number of parameters. As a result, NS equations were further transformed in terms of dimensionless numbers for instance Reynolds number, which further allows neglecting lesser important terms and NS equation reduces to:

$$\frac{\partial \mathrm{P}}{\partial \mathrm{x}} = \frac{\partial}{\partial y}\left(\eta \frac{\partial u}{\partial y}\right) \tag{1}$$

$$\frac{\partial \mathrm{P}}{\partial \mathrm{z}} = \frac{\partial}{\partial y}\left(\eta \frac{\partial w}{\partial y}\right) \tag{2}$$

$$\frac{\partial P}{\partial y} = 0 \ \Rightarrow P = f(x, z, t) \tag{3}$$

By successively integrating the equation above, and applying boundary conditions, we get the expression for the velocity.

$$u = -Y\frac{(h-Y)}{2\eta}\frac{\partial \mathrm{P}}{\partial \mathrm{x}} + u_0\frac{(h-Y)}{h} + u_h\frac{Y}{h} \quad (4)$$

$$w = -Y\frac{(h-Y)}{2\eta}\frac{\partial \mathrm{P}}{\partial z} + w_0\frac{(h-Y)}{h} + w_h\frac{Y}{h} \quad (5)$$

Boundary conditions for velocity are defined as Y = 0 → $u = u_0$ and Y = h → $u = u_h$. Subscripts h and 0 are references to upper and lower surfaces conditions respectively. Therefore, u_h, v_h and w_h are upper surface velocity components in x, y, and z-direction respectively and u_0, v_o *and* w_o are the velocity component of the lower surface. Figure 3. shows a typical model which is used to derive Reynold's equation, where AB is the top stationary plate of length L inclined with some angle α to make a wedge. Bottom plate CD is moving at a velocity (u) in negative x direction.

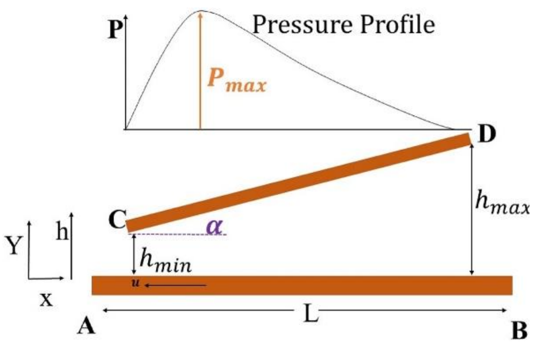

Fig. 3. Reynolds model.

For deriving Reynolds' equation, flow rate calculated from velocity was used together with the continuity equation. Equation (6) shows the generalized Reynold's equation developed for lubrication pressure approximation, here h is the film thickness shown in figure 3.

There is some significance to each term of the above equation (6). The first two terms of the equation show the Poiseuille flow, and the second pair of terms known as Couette flow and refers to the flow rate due to surface velocities. The Reynolds equation became a cornerstone in the history of lubrication, which is derived with some assumptions [29] outlined as: (i) body forces are taken as negligible, (ii) over the film thickness there is constant pressure ($\partial P/\partial y = 0$), (iii) at the boundary of solid surface no-slip condition is assumed, (iv) laminar flow of lubricant (Reynolds number is low), (v) As compared to viscous forces both surface tension and inertia forces are insignificant, (vi) both velocity gradient and shear stress have significance only across lubricant film, (vii) Newtonian lubricant is considered, (viii) Boundary surfaces of lubricant are relatively either at a small angle or parallel to each another.

$$\begin{aligned}\frac{\partial}{\partial x}\left[-\frac{\rho h^3}{12\mu}\frac{\partial P}{\partial x}\right] + \frac{\partial}{\partial z}\left[-\frac{\rho h^3}{12\mu}\frac{\partial P}{\partial z}\right] & \\ + \frac{\partial}{\partial x}\left(\frac{\rho h(u_h + u_{0)}}{2}\right) & \\ + \frac{\partial}{\partial z}\left(\frac{\rho h(w_h + w_{0)}}{2}\right) & \\ + \rho(v_h - v_0) & \\ - \rho u_h\frac{\partial h}{\partial x} - \rho w_h\frac{\partial h}{\partial z} & \\ + h\frac{\partial \rho}{\partial h} = 0 & \end{aligned} \quad (6)$$

In the 1960s researchers have managed to numerically solve Reynold's equation while considering the variation of fluid properties [30-38]. Additionally with energy equation to consider the temperature variation [39]. Later in 2003 Spikes extended the Reynolds equation for allowing the fluid to slip at the solid surfaces [40]. Szeri presents an extensive review of relaxation of Reynolds assumptions in six different fields namely, Elastohydrodynamic (EHD) effects, Turbulence effects, Inertia effects, Thermo-hydrodynamic(THD) effects, Non-Newtonian effects, and Compressibility effects [41].

Reynolds equation has also been modified for very high contact pressure (EHL) Rajagopal et. al. [42] demonstrated the inconsistency in the conventional Reynolds equation. As a result, an equation of motion to incorporate variation of viscosity (Barus Pressure viscosity formula) was derived and found an additional term in the modified equation. Numerical simulation results from this new equation show a slightly higher pressure compared to traditional Reynold's equation at a much higher viscosity for a cylinder on plane (rolling contact), both elastic and rigid cylinders problem.

$$\frac{d}{dx}\left[\left(\frac{h^3}{\eta}-12\alpha\int_0^h y(h-y)\frac{\partial u}{\partial x}dy\right)\frac{dP}{dx}\right] = 6U^*\frac{dh}{dx} \quad (7)$$

Where α = Pressure viscosity coefficient in Barus Formula, U^*= Characteristic velocity. If α=0 the modified equation will reduce to the conventional Reynolds equation.

The above-described model is further modified by Bayada, Cid et al. (2013) [43]. He incorporated the additional cavitation model, avoiding the simplification hypothesis used in Rajagopal's model [42]. In 2019 Almqvist et. al. [44] shows that for compressible fluids inertia terms always may not be neglected. According to him, compressibility determines whether inertia can be neglected or not. He followed the equivalent asymptotic analysis as Rajagopal's for density variation due to pressure instead of viscosity variation due to pressure.

If we consider the contacting surfaces are smooth or $\lambda > 0.5$ (lambda), then Reynold's equation and full model including film thickness, variable viscosity, energy model, shear-thinning model give satisfactory results and are well understood [18,36,45-50]. However, several questions regarding mixed lubrication asperity contact are yet to be resolved. For example, what happens when thickness of lubricating film is reduced to within nanometer range, or nearly zero. Does the Reynolds equation, and its assumptions, remain valid? Or we need some other equation at that place to define the flow, which can account for the continuity of flow around the asperity contact. Many authors have used a 2D model to numerically simulate mixed lubrication [3]. Ideally, ML problems involving asperities contact cannot be simplified to 2D. The simplification in 2D is only possible when the surface roughness is ignored or modelled stochastically [18].

3. ROUGHNESS EFFECT IN LUBRICATION

Interacting surfaces roughness significantly effects flow conditions for example pressure fluctuations, stress distribution, thickness of film (clearance), friction, temperature and wear [51]. At a sufficiently small scale, we can observe that engineering surfaces are all rough, which has a direct impact on lubrication failure. Recent studies on heavily loaded non-conformal contacting surfaces have shown that the contacts with pressure distribution resulting from low-surface roughness frequency are more failure prone, however contacts with higher-frequency surface roughness are relatively less susceptible to failure [24].

In the past, both stochastic and deterministic models of roughness have been used to characterize roughness quantitatively. Stochastic models of rough surfaces dominated early studies of roughness modelling, which used a few statistical parameters to model rough surfaces [18]. Rough surface contact problem is described by using micro contact asperity models. Greenwood and Williamson (GW) presented one of the earliest analytical roughness contact models by taking into account an exponential or Gaussian asperity heights distribution [52]. This model was later extended to obtain the pressure condition between two spheres by considering that one of them is rough [53]. The GW model was proven more reliable than other models [54]. Greenwood and Tripp [52] expanded the model to study a pair of rough interfaces. In 1988 Zhu and Cheng [7] investigated the influence of surface roughness on EHL point contact by making use of Greenwood and Tripp asperity contact model in conjunction with Reynolds average equation. These models have not considered the deformation, and the Greenwood and Tripp model used simple and identical asperity geometry, which cannot handle practical conditions, but for initial studies, the models were good and easy to use. Patir and Cheng made a notable contribution to stochastic mixed lubrication models development [55]. Chow et al., [56] pointed out that the results obtained with stochastic models were restricted to a particular category of asperity orientation that is, peaks oriented either transversely or longitudinally. The stochastic models cannot deal with 3D surfaces. The results obtained with stochastic models give an overall consequence of surface topography and do not have the capacity to provide comprehensive information relating to high-pressure points [7], deformation of asperity, which are key factors influencing lubrication film breakdown, and surface failure mechanisms [3].

Over time, there was a need to define a deterministic model which could be robust where roughness can be defined at each node [22]. Therefore, some deterministic models were developed for steady-state conditions which cannot be used directly [3]. Initially, the study started in 1963 with simple typography which can be defined easily and more suitable for numerical solving methods for example irregularities in the form of a sinusoidal wave [57,58]. The Multilevel (multigrid) technique was introduced in the 1970s which accelerates the process of calculation based on the multigrid method and is used by Lubrecht [59] to develop algorithms, to numerically solve longitudinal and transverse roughness for EHL in two dimensions. In deterministic models, digitized geometric surface profiles are used a numerical simulation model inputs [33,60]. Presently, most modern surface measurement instruments can digitalize the surface roughness with high resolution, which produces a large amount of data. Consequently, it takes a longer time for computation.

To predict rough surface influence on Lubrication, we must deal with several challenges. For example, in order to capture all information in the random roughness texture, an extremely high degree of discretization is required [61]. A second problem is to calculate micro deformation, thermal, and non-Newtonian effect sensitively. Then additional transient effects complicate the calculation. Several studies have examined the behaviour of roughness in EHL [8-15,62]. The initial study began with a study of harmonic waviness kind of roughness under contact, which helped to achieve quantitative predictions of the influence of wavelength, amplitude for Newtonian lubricants [24,45]. However, the Newtonian models give a large overestimation of the friction coefficients [14]. Chapkov, Venner et al. [14] used the non-Newtonian (Eyring model) model to forward this study. He also derived a general model that can include both kinds of behaviours. Another part of the roughness influence study is the effect of roughness orientation on EHL contact [63]. This area of research began in the 1970s, but it has become more important recently in finding the optimal surface patterns for the particular operating condition, for example, micro dent on the surface works as a reservoir to effectively lift off the real surface features [64].

In 2006 Hooke suggested a simplified method to deal with the roughness effect in EHL contact [65,66]. His analysis of the sinusoidal roughness effect in EHL contact suggests that roughness amplitude reduces due to loading (attenuated) and generates a deteriorating complementary wave under the conjunction at the inlet. Which enters with an entrainment velocity at the contact, and the wavelength of this complementary wave is determined by the relation $\lambda u/v$, hence λ is the roughness wavelength (original), u is entrainment velocity and v is the roughness velocity. Perturbation analysis has been employed to accurately calculate film thickness and corresponding pressure with low roughness amplitude and compares the results to those obtained with a simple multigrid solver and experimental results [50]. Later Hooke et al. [50] extended this method for calculating thickness of film and pressure within rough contact in EHL. Using Fourier transform techniques, we can directly calculate attenuated roughness and their associated pressure ripples. This analysis, however, can only be applied where roughness's amplitude is far lower than the film - thickness. While the theory of amplitude attenuation has not been developed for mixed lubrication, it may prove useful in predicting roughness behavior with lubrication regime shifts. Hence, the roughness attenuation approach can help engineers to reduce the risk of machines operating under mixed lubrication if they are planning to design & operate them under EHL conditions [48].

4. MIXED LUBRICATION

Mixed lubrication refers to the interaction of two surfaces interfacing an ultra-thin lubricant layer and asperity contacts that can cause fracture and elastic-plastic deformation of interacting surfaces [67]. It is impossible to avoid mixed lubrication in engineering applications. Therefore, it is one of the most challenging area in lubrication science to model and predict the behaviour of mixed lubrication contacts. Significant work has been conducted in this area [22]. The first early theoretical description of mixed lubrication has been presented by Christensen in his quantitative mathematical model [68,69] where surface roughness with random height distribution has been modelled without giving consideration to the surface deformation [70]. He presented the film thickness equation in two parts $(h + h_s)$ where, h is a smooth surface and h_s is a result of surface roughness and has been measured from/at nominal level [71], and the mean value of h_s always zero over the surface. Subsequently, he gave the mathematical model of mixed lubrication in sliding contact [68]. In his model, he assumes that the formed film thickness will be in a form that Reynold's equation will always be valid in that area. Therefore, this model binds itself to Reynold's assumptions [72]. Following from this model an array of models were proposed to study roughness effects on hydrodynamic lubrication [25,73,74].

Later a new concept arises, that the modelling of mixed lubrication should be a contribution of the hydrodynamic and contact mechanics' model. Such a model was first introduced in 1972 by Johnson et al. in superposition principle [75]. In this work, Greenwood and Williamson (GW) model is used to configure the load supported by asperity contact and conventional theory of lubrication of smooth surfaces given by Dowson and Higginson [52]. To date, most models of surface contact assumed that only one surface was rough in contacting pairs, which was extended by assuming both rough surfaces [76]. Johnson, Greenwood et al. [75] used the GW model for presenting roughness in their modelling of mixed lubrication. Although the model is very simple in its implementation which is the main reason for its popularity. However, the GW model has two disadvantages. One is the spherical summit of constant radius of curvature which is an unrealistic representation of surface topography and the other is the Hertzian theory is used for elastic displacement of each asperity, without considering the interaction between the asperities [77]. Tsao and Tong [78] used the combination of Reynolds equation and the solution from boundary lubrication to calculate the load and friction of the lubricated contact, where asperities are considered in cylindrical form with a spherical tip and the height of the with an assumption that the cylindrical asperity exhibits gaussian distribution. Later in 1978 Patir gave the deterministic model for random generation of 3-dimensional roughness with predetermined properties [79]. Patir and Cheng (PC) utilised this model to introduce the Average Flow Reynolds equation with the multiplication of shear flow factor and pressure attributes in the conventional Reynolds Equation [55]. Both the factors are a function of surface height and are determined by using numerical flow simulation. Average gap is utilised to calculate, average contact pressure and average flow model, are based on undeformed surface roughness, which can give significant error in gap function and pressure in the low lambda λ region [80]. They have also used this model for sliding contacts for numerically generated rough surfaces [81]. However, this model cannot handle non-isotropic roughness in real engineering surfaces [82]. Generally, asperity contacts (solid contacts) are treated under two different approaches in deterministic lubrication modelling: (1) The lubricant film and solid contacts are treated separately [3,80], solid-contact and fluid-film pressure variables have been defined which are sequentially updated in an iterative process: (2) A unified system of equations deals with solid-contact regions and lubrication [33,83,84], in which old and new pressure arrays are updated during each sweep, covering both lubrication and solid-contact regions. Recently, Liu proposed a set of Lubrication contact Boundary Conditions (LCBCs) and solved a simple geometry with flow blocked by solid square and cylinder. The results show a higher pressure build-up compare to traditional solutions [34]. In this review, various mixed lubrication models with mathematical equations and assumptions have been discussed. Most existing models use Reynolds' equation or its modified form with varying approaches to solid asperity contact (described in the respective sections). Generally, well-established Finite difference methods have been applied for solving Reynold's equation (partial differential equation) and the procedure can be found in related books [59]. In the following, several mixed lubrication models and related areas are briefly described.

4.1 Average flow model

In the average flow model, for solving Reynold's equation in terms of average pressure between two rough interacting surfaces, film thickness is defined as the thickness of the local film is an addition of nominal film thickness h and wall roughness δ_1, δ_2 which is a random roughness amplitude measured from the mean level (see figure 4). Wall roughness parameter (δ_1 and δ_2) assumed to have Gaussian distribution of height with zero mean and standard deviations σ_1 and σ_2 with combined root mean square value $\sigma = \sqrt{{\sigma_1}^2 + {\sigma_2}^2}$. Surface roughness effect depends on the value of lambda ratio ($\lambda = h/\sigma$). For $\lambda < 3$ partial lubrication regime has been defined where asperities start interacting.

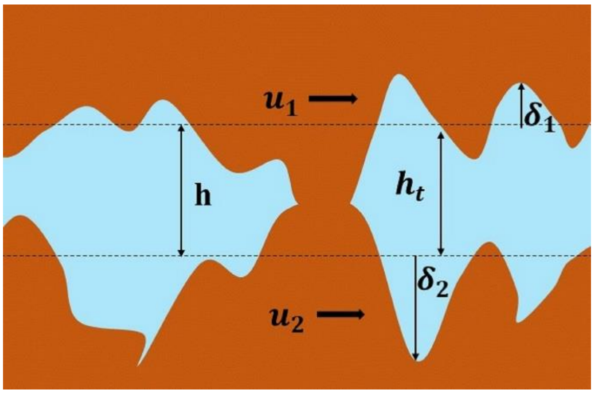

Fig. 4. Film thickness representation in average flow model.

$$\frac{\partial}{\partial x}\left(\phi_x \frac{h^3}{12\eta}\frac{\partial \bar{P}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\phi_y \frac{h^3}{12\eta}\frac{\partial \bar{P}}{\partial y}\right) = \frac{u_1+u_2}{2}\frac{\partial \bar{h}_t}{\partial x} + \frac{u_1-u_2}{2}\sigma\frac{\partial \phi_x}{\partial x}+\frac{\partial \bar{h}_t}{\partial t} \quad (8)$$

$\phi_x, \phi_y and\ \phi_s$ have the property:
$\phi_x, \phi_y \rightarrow 1$ $\quad \phi_s \rightarrow 0$ $\quad$ As $\frac{h}{\sigma} \rightarrow \infty$

Consequently, the above average Reynolds equation will reduce to classical Reynold's equation in terms of smooth surface as film thickness becomes larger with respect to roughness (RMS). The pressure flow factors ϕ_x, ϕ_y can be obtained through numerical simulation explained in his paper which are very complex themselves [85].

Later in 2004 Wang, Zhu et al. [86] proposed a new interactive deterministic-stochastic modelling approach similar to the unified numerical approach (Macro-Micro approach). Using the Patir and Cheng (PC) [55] average flow model, the flow and asperity contacts are modelled separately, namely, a flow model for lubricant dynamics and an off-line contact model for asperity pressure. Deformation is also considered in two parts: the deflection of the centre surface and the deflection of the asperity with respect to the centre surface. For more information, readers are referred to reference [86].

Goglia et al. used the deterministic model of rough surfaces (wavy surface), to evaluate the roughness of surface impact on EHL line contact in sliding for steady-state with assuming a continuous oil film always exists between the contacting surfaces [87]. As a result, this analytical model is not applicable for mixed lubrication. They first gave a model with a single asperity effect on pressure, sub-surface octahedral shear stress and film thickness [87] and then calculated the effect for roughness in the form of a sinusoidal profile with a defined wavelength and amplitude.

Zhu and Chang, in 1988 [7] utilized the average Reynolds equation which was previously presented by Patir and Cheng [68], incorporating both the relationship between pressure-viscosity and the elasticity equation. The model separately calculates contact and hydrodynamic pressures and then simply superimposes these to balance the applied load [33]. In this model asperity, the contact pressure is computed by the relationship which was developed by Greenwood and Tripp [76] who considered Gaussian distribution of surface asperities. It was shown that the roughness parameters resulted in notable effects on load sharing ratio and contact asperity pressure. Moreover, these results are in synch with Patir and Chang results. However, the important issues in this literature are, roughness has gaussian distribution, which is not true in engineering practice, to make the shear flow factor zero the roughness on two opposite surfaces was modelled with the same structure and Root Mean Square (RMS) values. Additionally, this model was not good for high loading conditions. Therefore, this work is advanced, with the help of various numerical solution techniques (Newton-Raphson iterative method) for overloaded line contact and presented the roughness impact over lubricant film at the exit and pressure spikes [62].

Later, the flow factor method is also extended to non-Gaussian surfaces. For different alignment directions of roughness, Morales-Espejel [88] proposed a simple analytical approach to derive flow factors by including third-order terms in approximation. The resulting Reynolds equation gives a good prediction of the global impact of roughness on film thickness.

It is necessary to digitize surface topography accurately to simulate mixed lubrication numerically which can be computationally expensive [33]. Reducing the number of grid points may, on the other hand, lead to unrealistic results. Similar to the well-known average flow factor model [55] Sahlin, Larsson et al., [70] in 2010 developed a systematic method to calculate flow factor for measured surface topography. The flow factor is calculated with the help of homogenization and calculation of roughness deformation has been performed with Discrete Convolution-Fast Fourier Transforms (DC-FTT). As part of homogenization, local roughness scales are separated from global smoothness scales. With only a topography measured for a small part of the surface, it is possible to solve local cell problems containing all the information necessary to determine how roughness contributes to local surface deformation and flow. This model provides a detailed flow description for cross pattern surface roughness. Consequently, results are incomparable with the PC flow factor however for longitudinal roughness both the models are comparable. Overall, this is a fast and straightforward method that takes real roughness as input [70]. Most of the ambiguities of the earlier average flow model are eliminated in this new model. However, this model gives average results based on flow factor is not able to provide flow conditions around the asperity contact.

In 2008, the Lattice Boltzmann Method (LBM) was used to determine the flow factor in tribo-models based on extended Reynolds theory suggested by Patir and Cheng [89]. This study examines the impacts of bulk geometrical parameters and Reynolds number on flow factor and velocity distribution for a Newtonian fluid of constant viscosity.

Overall, these findings were not very significant in terms of numerical simulation of mixed lubrication because information regarding asperity deformation, fluctuations in local film thickness and local pressure peaks are difficult to be deduced. To study the breakdown of lubrication and subsequently surface failure, these characteristics are crucial [90]. However, these results were helpful to understand roughness overall effect on lubrication but cannot handle real mixed lubrication.

Surface modification and lubrication are key elements for managing friction of tribo-surfaces like piston assemblies, seals, rolling bearings, and gears. PC flow model together with Greenwood and Tripp model have been recent, utilized to solve mixed lubrication problems. The generated numerical results are compared with experimental results which shows qualitatively good agreement [25].

4.2 Some independent models for mixed lubrication

A 2D model of mixed lubrication introduced by Chang et al. [3] is a deterministic model of tribo-contact. Pressure is calculated by this model in two parts: first part uses already developed transient EHL model for smooth contact (P_f) with considering non-Newtonian shear-thinning effects and S in the Reynolds equation is Eyring modifying factor; in the second part, if asperity contact has occurred, pressure (P_s) is calculated using asperity Contact Equation given below by considering asperity deformations are elastic, and the contact is frictionless.

For $h(x,t) > 0$

$$\frac{\partial}{\partial x}\left(\frac{\rho h^3}{12\mu} S \frac{\partial P_f}{\partial x}\right) = u\frac{\partial}{\partial x}(\rho h) + \frac{\partial}{\partial t}(\rho h) \qquad (9)$$

The below integral equation (10) governs the asperity contact pressure. Right-hand side of the equation (10) yields surface integration at x location in an asperity contact point in the absence of asperity pressure. This interpenetration is invalidated by asperity pressure that generates surface deformation in accordance with deformation integral on left-hand side. When variables satisfy all governing equations only then a solution to the problem can be obtained. Overall, the theoretical background of this model violates all sorts of rules. This model ignores the Reynolds equation at the time of asperity contact, which violates continuity of flow.

For $h(x,t) = 0$

$$-\frac{1}{\pi E'}\int_{x_i}^{x_0} P_s(s,t)[\ln(x-s)^4 + c]\,ds = -\left\{-\delta(t) + \frac{x^2}{2R} + r(x,t) - \frac{1}{\pi E'}\int_{x_i}^{x_0} P_f(s,t)[\ln(x-s)^4 + c]\,ds\right\} \qquad (10)$$

As a result, boundaries are closed and there is no flow in the area right after and before the asperity contact. In other words, the model in this article uses a bump that shows a transverse direction change from negative infinite to positive infinite, indicating a solid wall where the asperities are in direct contact, which is an erroneous interpretation of the asperity contact. As the thickness of film tends to zero at the asperity point of contact, it is important to investigate the cause of pressure buildup at the contact point.

In 2004 Bakolas, V, and Mihailidis, A. [91] simulated mixed lubrications contact of an FZG type C gear pair for known roughness profile (deterministic model). In his analysis of mixed lubrication, he considers asperity contact, asperity interaction, and lubricant flow to calculate surface deformation due to pressure build-up between rough contacting surfaces. He considered the deformation due to asperity pressure and fluid pressure in calculating the deformation of solid roughness peaks. Moreover, thermal, sliding and non-Newtonian effects are also considered by using power law. Additionally, subsurface stress fields generated due to pressure and friction coefficient are also calculated for the FZG type C gear pair. This model is derived on the basis of superposition of a smooth fully flooded and a dry rough contact [91].

Jiang in 2005 presented a very basic model for pressure, film thickness and generated temperature calculation while numerically modelling a single asperity at the EHL contact. An Ellipsoidal Bump has been used over a stationary surface to see the effect of asperity interaction in EHL point contact for different ellipticity parameters. The other surface is smooth and moving with a velocity. This single asperity produces a local high pressure, dangerously thin film, and high temperature, which causes partial depression of the single bump. However, the bump is considered on the stationary surface. In this model, the pressure field is calculated by using multilevel method whereas elastic degradation is assessed by using multilevel multi-integration methodology [13].

4.3 Unified numerical approach

Zhu and Hu in 1999 [33] presented a new unified approach for numerically simulating lubrication for point contact in all regions from Elastohydrodynamic, mixed and boundary lubrication. This model extends the traditional EHL model for mixed lubrication by applying threshold conditions for the asperity contact. The value of the threshold can be specified with respect to the dimensions of lubricant molecules [84]. To mathematically model the asperity-contact, where oil film disappears (lower to a threshold value $h < h_{min}$), they made pressure flow (Poiseuille Term) and squeeze term zero in Reynold's equation for 2d roughness. Hence the basic idea of this model is that use the full Reynolds equation (11) for hydrodynamic lubrication and at asperity contact reduced form of equation (12) is used where the Pressure and squeeze term are zero. Thus, they can simulate lubrication across the entire domain using a single Reynolds equation. The equation is solved with film thickness equation and to model mixed lubrication conditions a minimum film thickness is predefined such as 10 nm (threshold value) to assume the asperity contact condition.

Reynold's equation that governs pressure in the hydrodynamic region:

$$\frac{\partial}{\partial x}\left[\frac{\rho h^3}{12\mu}\frac{\partial P}{\partial x}\right]+\frac{\partial}{\partial z}\left[\frac{\rho h^3}{12\mu}\frac{\partial P}{\partial z}\right] = \mathrm{U}\frac{\partial(\rho h)}{\partial x}+\frac{\partial(\rho h)}{\partial t} \qquad (11)$$

As the film thickness gets reduced to a predefined threshold value, asperity contact is considered, and Reynold's equation will be reduced:

$$\mathrm{U}\frac{\partial h}{\partial x}+\frac{\partial h}{\partial t}=0 \quad at \quad h=0 \qquad (12)$$

This approach is further used [92] to determine the impact of surface topography and its orientation over film thickness within circular contact. The literature reports that the unified numerical technique is able to handle measured rough surfaces subject to harsh operational conditions in turn can simulate transition starting at full film EHD to mixed lubrication and down to boundary lubrication [92]. C H Venner examined the same problem with the exact same conditions and grid size in 2005 and concluded that the film thickness cut-off resulting in contact is due to a very large error in discretization. He also revealed that the results produced by a unified numerical approach may not be physically meaningful [93].

Dong Zhu [32] show the roughness effect in mixed lubrication by using a unified numerical approach. The author analysed various cases, with 3 types (shaved: isotropic topography, ground: transverse, longitudinal) of measured machined roughness range of 0.0 and 0.1 up to 1.5 micrometres and rolling velocity range cover 15000 mm/s to 0.001

mm/s for transient condition. In all cases one surface is smooth and the other one is rough. These results demonstrate noticeable impacts of roughness topography and orientation on the contact load and area ratio and this effect is very less in EHD and BL. All parameters such as contact load ratio, maximum pressure, friction and flash temperature including sub-surface shear stress significantly increase with an increase in roughness. However, according to Venner [93,94] Using the 257*257 grid size again, these results are questionable in terms of accuracy for a unified numerical approach. Later in 2016 unified numerical approach is extended to take account of the thermal effect in all the lubrication regimes. Hence, enhanced unified TEHL model applied to point contact problem is proposed while considering 3D real roughness features [95]. The unified numerical approach is also applied for calculating asperity contact in gear working under mixed lubrication regions [96].

Deolalikar et al. developed a model to calculate pressure and film thickness which includes roughness interaction with transient conditions for elliptical rolling/sliding contact for Newtonian lubricant [97]. Further generated results were utilized to calculate heat generated and temperature rises due to roughness peaks friction within the contact zone.

Cavitation zone is underestimated by the General Reynolds equation for lubrication analysis, which yields inaccurate results. A mixed lubrication model for point contact by Zhang et al. in 2012 was presented by incorporating Jacobsson-Floberg-Olsson mass conservation boundary condition based on Elrod's algorithm. By using a unified numerical approach, this model accounts for starvation and local cavitation in the contact zone, and a 0.5 nm threshold value is set for asperity contact [84].

5. CFD APPROACH

Calculating pressure film thickness from Reynold's equation and elastic deflection using Hertzian contact theory, require users to develop their own code. However, it gives reliable results for the converged solution. The recent fast development in computer software, allows solution of complex equations with a finite volume approach, used in Computational Fluid Dynamics (CFD). CFD employs full Navier-Stokes' equations, to predict the flow behaviour around the complex geometries. This approach allows the possibility of expanding the computational domain, and users can see the variations in other parameters, for example, the entire fluid domain can be modelled in 3D and allows to resolve all gradients inside the geometry. Several authors have used the CFD approach in EHL problems using CFD codes [11,15,38,98-105]

Firstly, in 1988 Chen et al., [98] justified the Reynolds assumption with the CFD model and demonstrated the applicability of CFD for hydrodynamic lubrication. Almqvist et al. showed that the Reynolds and CFD approach gives approximately similar numerical results with stability [11]. Hartinger et al. [15] compared the isothermal CFD results with Reynold's theory and found nearly equal results except for high viscosity and velocity rolling/sliding conditions. With the CFD approach, he showed that pressure across the film thickness varies for EHL contact, which is contrary to the Reynolds assumptions. To evaluate the driving force behind the pressure variations he used the momentum equation. Gertzos et al. have worked on lubricated journal bearings in a Newtonian lubricant and a Bingham fluid to study performance characteristics [100]. Bruyere et al. have employed the Finite Element Method for solving Navier-Stokes' equations incorporating elastic deformation [106]. In which a non-Newtonian rheology model was employed alongside thermal and compressibility effects. Gao et al. [104] developed a 3D CFD model for plane water lubricated bearings applied in hydrodynamic lubrication. Parallel and perpendicular velocity gradients to the flow of lubricant change, in the EHL line contact problem, for rough surfaces [82]. Velocity gradient along the fluid direction is effected by the peaks of rough surfaces. Hajishafiee et al. [105] modelled the EHL rolling/sliding contact with Fluid Solid Interactions (FSI) which is stable over different contact conditions while considering compressibility, cavitation and thermal effects.

The CFD approach uses momentum and continuity equations simultaneously, which provide enhanced flexibility in terms of advanced modelling to accommodate rheological modifications. In this method, the velocity is calculated from the momentum equation and iteratively corrected by the pressure equation. Therefore, if mass continuity is our priority full Navier-Stokes' equation should be performed over Reynolds's equation. In the CFD approach efficiency is significantly reduced because of the extended size of the problem which is a major disadvantage. In contrast, in tradition Reynolds's approach, Reynolds's equation and film thickness equations are joined to solve film thickness and at the same time its corresponding pressures. If acceleration of the solving process is intended, then multilevel techniques could be adopted. In the Reynolds approach, computer codes must be generated using Reynolds' equation and, if the thermal simulation is required, it is necessary to solve energy equations simultaneously, making the simulation process more complicated.

6. MOLECULAR DYNAMICS (MD)

Generally, thin-film approximations fail due to continuum mechanics assumptions, which become invalid when lubricant thickness at interfaces is comparable to molecular size [107]. However, linear relationship between load and contact area appears valid for dry contact [67,108]. The study of fluid flow at the microscale can be carried out by utilizing molecular dynamics, which is a computational method based on simulating the dynamics of individual molecules [109]. Atoms or molecules in MD simulation system obey classical mechanics laws. To obtain atomic trajectory MD requires solving the equation of motion. This way MD simulation can help to examine and analyse the unobservable tribological aspects that occur at the nanoscale at the contact. Many authors have employed MD simulation to analyse friction and wearability evaluation [110,111]. Using MD simulation, Ghaffari et al. studies the effects of temperature and the mixing of nanoparticles with lubricant oil on friction [112]. Zheng, Zhu et al. (2013) have studied the roughness effects on lubrication using MD simulation and n-alkanes (hexadecane) as a lubricant. This study helped to better understand mixed lubrication and sheds light on confined flow behaviour under normal load within interfacial contact. They identified that the lubricant quantity plays an important role to determine contact area. They found out that subject to a particular load and surface roughness the creation of a cavity occurs, when lubricant molecules fill up that cavity, it will start to support bearing load which results in a decreased contact area [67]. In 2013 ultrathin lubrication reaching to dry friction, MD simulation was conducted, subject to high shear and pressure. Later in 2018, a hierarchical multiscale technique was suggested for examining rolling contact fatigue incorporating lubricant effects. MD simulation is used at the nanoscale for modelling lubricant and calculating coefficient of friction. And to predict fatigue life using contact stress analysis finite element method is utilised. In spite of the fact that this model was developed for smooth contacting surfaces, it is suggested that it can be extended to rough surfaces as well [107].

7. EXPERIMENTAL

Numerous numerical and experimental research work have been conducted in the past decades to understand the topography effect within EHL contacts. Mainly two techniques have been utilized in an experimental study of lubrication namely, surface force apparatus and optical interferometry. For additional information regarding the techniques, readers are suggested to reference [113-115]. Numerical and analytical models in mixed lubrication lack experimental validation [113,116]. A majority of experimental validations are restricted to the study of the effects of individual and multiple irregularities on smooth surfaces, such as artificially created dents and roughness features. Due to advancements in optical interferometry techniques, now it's possible to measure ultra-thin film thickness [48]. The influence of transverse and longitudinal bumps on point contact was studied by Kaneta, Sakai et al. (1992) [114] using optical microscopy to validate their numerical results. later in 2003 Félix-Quiñonez, Single ridge in an EHL point contact was reevaluated by employing a high-speed colour video camera aided by interferometric technique by Ehret et al. [115]. In 2005, this experimental work was extended by examining the influences of square-shaped surface texture on 3D flat-top travelling over EHL point contact, and its numerical solutions were compared to these experimental results. In order to obtain the complete history of defects deformation, optical interferometry was combined with a high-speed, colour video camera. Additionally, image analysis was also used to produce a map of film thickness across the total contact area. Surface deformation in high-pressure regions was dependent on the capability to trap lubricant passage through inlet zone in groove. These results helped to understand and predict the effects of real roughness features [117]. Thin film phase-shifting interferometry and colorimetric interferometry was used for examining impact of real surface roughness patterns within mixed lubrication region. In 2008 dents of varying depth and position are passed through lubricated non-conformal contact. High pressure spikes are observed at indentation edge in the sliding direction. Film thickness was influenced by sliding to rolling ratio. It was suggested that the micro dent can be viewed as a lubricant reservoir, which emits up and downstream lubricant which was dependent on sliding-rolling ratio. The lubricant released by micro dents lifts off the original surface features [64,118].

Start-up operation in mixed lubrication region can cause surface damage due to asperity interaction. An investigation was conducted for rolling/sliding conditions to assess the impacts of roughness (artificial or real) during initial running in of nonconformal contacts on the formation of mixed lubrication film [119]. This study on mixed lubrication shows significant pressure fluctuations due to local asperity interactions. Asperity contact, where the film has collapsed completely, is not easy to simulate numerically, however, with optical interferometry, the detail changes in film thickness could be observed. Hence, Krupka Sperka et al. (2016) combined the Amplitude attenuation theory with experimentally obtained measurements. A comparison of experimentally observed roughness deformations and complementary wave effects with numerical results shows good agreement. [48]

8. SUMMARY

The Mixed lubrication region is common in various rotating machine components, like gears and low speed rolling bearings. Moreover, the ML regime will become increasingly significant in future due to the current (efficiency increasing) trend of reducing the lubricant film thickness e.g., in rolling bearings. Reynold's equation has been utilized for many years for predicting pressure distribution in EHL and full film lubricated contacts. During this review, we intend to test whether Reynolds Equation holds true in mixed lubrication scenarios when almost zero film thickness is experienced. A brief description of Reynold's equation is given in the beginning, along with assumptions for smooth contacts.

Surface roughness is known to have an immediate and significant impact on flow conditions, resulting in increased pressure fluctuations, variations in film thickness (clearance), friction, heat generation, wear, and adhesion. Various models have been proposed for predicting surface features' effects, which are discussed along with their benefits and shortcomings. The main improvement in mixed lubrication numerical simulation is outlined and presented in this paper. Modelling the micro geometric variation associated with the transient effect is the most challenging aspect of mixed lubrication. This is computationally intensive and takes a long time to solve. By implementing new techniques and using computational methods, these difficulties have been solved in the past.

Initially, the authors tried taking out the overall effect of roughness such as using flow factor. Many authors have proposed solutions to simplify the Reynolds equation (for example, a unified numerical approach), but their evidence has not been convincing, and their mathematical, numerical, and theoretical models have shortcomings. The other information, which is available essentially represents the approximate roughness solution, but it is underestimated over the whole field. However, all models work towards improving knowledge of mixed lubrication. Nevertheless, there is no model that can address the 3D problem of mixed lubrication asperity contact, which can define flow conditions around the asperity contact when film thickness reduces to zero.

This paper provides some insights into mixed lubrication by discussing some new methods recently developed for example Molecular Dynamics, new models for calculating flow factor (Lattice-Boltzmann method, semi-analytical method), roughness attenuation theory, and computation fluid dynamics. Additionally, some experimental validation of numerical models is also discussed. In recent years, optical interferometry techniques have enabled film thickness measurements, which can be combined with quantitative data derived from numerical models. These data can be used to develop a detailed but comprehensive model for mixed lubrication. An improvement in molecular dynamics simulation by including asperity contact with a developed EHL model can determine microscale phenomena that cannot be numerically simulated.

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