3D Printing Hypercomplex Fractals

Emma Fogarasi and Oleg Fryazinov
The National Centre for Computer Animation, Bournemouth University, UK

Abstract

This paper describes a method to efficiently define and 3D print unique kaleidoscopic hypercomplex fractals. The proposed method allows the user to easily edit fractals in three ways: changing the power the fractal is raised to, changing the number of sides of the kaleidoscope, and the angle and position of the fractal, which creates different shapes due to the kaleidoscope. In the example of the Mandelbulb and the inverted Mandelbulb fractals, where the shape of fractals is manipulated through the "kaleidoscopic effect", we create 3D fractal sculptures that can be realised as physical objects by using 3D printing. The flexible parametrisation of the fractal definition implemented as the Houdini tool allows the definition of many unique shapes of kaleidoscopic hypercomplex fractals that can be digitally fabricated with a little pre-processing. The paper results can be used for creating fractal-based sculptures as a part of 3D fractal art.

Introduction

Although the term "fractal" was introduced in the 20th century, self-similarities in geometry were well-known long before. The mathematics of fractals is not being considered mainstream; however, fractals consistently are used in art. Artists can use their infinite recursion to create beautiful, artistic forms that have made a basis for "fractal art", which describes primarily computer-generated art that uses fractals in its imagery. Recently the hypercomplex fractals started to appear in feature animation films and other visual media. It is worth mentioning that fractal art is predominantly 2D, as the algorithms that create fractals in 3D are hard to manipulate.

This paper explores hypercomplex fractals and presents the method to create fractal sculptures with unique shapes. By extending the definition of the 3D Mandelbulb algorithm together with the subsequent self-defined kaleidoscopic effect, we allow defining many various fractal sculptures with allowing for artistic control at the same time. The implementation of the algorithm as a Houdini tool allows the user to create a unique shape that can be converted into a format suitable for 3D printing. We have 3D printed several models to demonstrate the applicability of the algorithm. The presented pipeline allows the hypercomplex fractals to be available as an artistic tool to be used for other artists working with fractal geometry.

Background and Related Works

Fractals are infinitely complex patterns that are self-similar on different scales. This recursiveness appears in nature, such as mountains, coastlines, clouds[1], and even blood vessels. Mathematical definitions for different fractals can vary. This paper focuses on fractals that use complex and hypercomplex numbers in their definition, specifically the 3D Mandelbulb created from the 2D Mandelbrot set.

Mathematics of fractals. The Mandelbrot set is made by iterating a simple recursive rule using complex numbers where the pattern itself appears on the complex plane. It was first discovered by Robert W. Brooks who drew it in 1978 for a study of Kleinian Groups [14]. The mathematical definition of the fractal set was done in 1980 by Benoit B. Mandelbrot. A Mandelbrot set is defined by the equation $z_{n+1} = z_n^2 + c$, where z and c are complex numbers [6], and the point is said to belong to the set if $|z|<2$ for all iterations n. While the Mandelbrot set separates the set of complex numbers into two subsets, later it was noted that the fractal nature could be better perceived if the fractal visualisation is augmented with colours. For example, the colours of the points are determined by the number of iterations required to reach $r_{\max} = 2$ where $r = |z|$. In

Figure one, the red points are the points that require 0 iterations, pink points require 1 iteration, violet points require 2 and blue points require 3 or more.

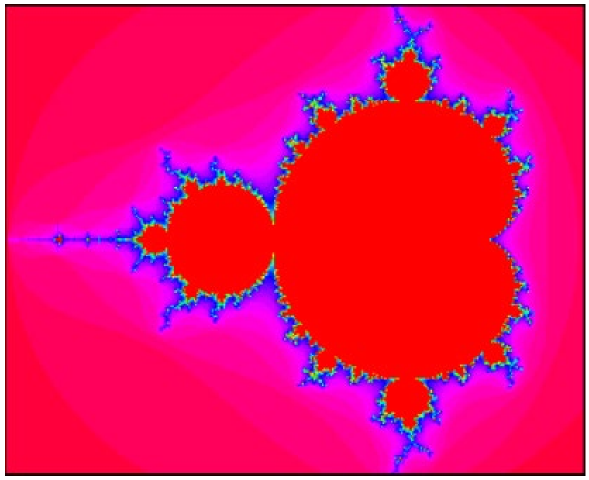


Figure 1: *The Mandelbrot set*

There are different ways to extend the Mandelbrot set from 2D into 3D. One of the ways of doing that is to rotate the set around its axis. The most well-known "3D Mandelbrot" is the Mandelbulb, a 3D shape discovered by Daniel White [7] where the extension from 2D into 3D is done by converting Cartesian coordinates to spherical coordinates, and then converting them back. The algorithm uses the definition similar to the Mandelbrot set, $z_{n+1} = z_n^2 + c$ however z and c here are hypercomplex (triplex) numbers representing 3D Cartesian coordinates. In its definition the exponential term is defined as

$$\langle x, y, z\rangle^n = r^n\langle \cos n\phi \sin n\theta, \sin n\phi \sin n\theta, \cos n\theta\rangle \tag{1}$$

Where $r = \sqrt{x^2 + y^2 + z^2}$, n is the order and angles are defined as $\theta = \text{atan2}\left(\frac{z}{\sqrt{x^2+y^2}}\right)$ and $\phi = \text{atan2}\left(\frac{y}{x}\right)$

While the Mandelbulb is a set of 3D points that define an object with a finite volume, it has an infinite surface area due to its infinite recursion. In this paper, the Mandelbulbs are frequently referred to as Hypercomplex fractals. This is because while the Mandelbrot set uses complex numbers, Mandelbulb uses Hypercomplex numbers which are similar to complex numbers but extended into 3D [13].

Fractal art. Fractals were used in architecture such as Gothic Cathedrals [15] and art long before the term "fractal" was introduced. Fractal art, however, is predominantly 2D with only several exceptions such as fractal terrains [1] and relief carvings [3]. The main reason to stick to 2D fractals for artistic purposes is that the algorithms that create fractals in 3D are hard to manipulate.

In recent years, hypercomplex fractals have become an integral part of visual effects. They have been used in films such as the Guardian of the Galaxy Vol 2 and Annihilation. In Big Hero 6 [4], a variation of the Mandelbulb algorithm was used with parameters that allowed the VFX team to easily create a large variety of 3 dimensional forms. Similarly, our Houdini tool takes variations of Mandelbulbs with editable parameters which then can be applied for 3D printing.

“Fractal Effervescence” by David April made in 2006 is an art piece that creates fractal patterns from mathematical formulas to create a visually impactful image. The artwork was created by fusing three image files which were made with the software Apophysis developed by Mark Townsend. The three files had different types of transformations which created this image. [11]

Johan Andersson used 3D fractals to create Surreal Fractal Jewelry and Accessories soon after Daniel White discovered the Mandelbulb in 2009 [9]. One of his works involves creating fractal art by 3D printing Surreal Chess pieces. [12] His work is based on hypercomplex fractals, converting mathematical fractal algorithms to create 3D sculptures.

Houdini. Houdini is a software that uses a procedural system to create artwork. It has a node-based workflow which is ideal for visual effects as it allows their users to create dynamic simulations, and it also allows the user to build custom nodes with VEX, a language based on the C language. It is suitable for our work as it is aimed for visual effects and its custom nodes allow us to quickly visualise mathematical algorithms. [18]

Method and implementation

As mentioned previously, one of the main goals of this research is to create 3D fractal art with the visual being the main component, at the same time allowing for efficient visualization of the resulting fractal shapes. Since a Mandelbulb’s surface area is infinite, we approximate the shape of the hypercomplex fractal by representing the object with a high-resolution volumetric representation. The first step, however, is to allow for modifications in the equations that would result in different shapes, at the same time by keeping the visual appearance.

Despite some flexibility in the definition of the Mandelbulb, where the variations can be achieved by modifications of the order n (see equation 1), it is still limited in resulting shapes. To increase the variations, we have modified the formulation of the Mandelbulb (the equation 1) in the scope of our experiments.

First modification to obtain so-called “inverted” Mandelbulb was by using $\phi = \operatorname{atan2}\left(\frac{-y}{x}\right)$. The resulting shape becomes drastically different from the original Mandelbulb, as we show in the Figure 2 (a) and (b).

To create an even larger variety of Mandelbulbs without inventing a new formula, we manipulated fractals by experimenting with the conversion of the coordinates from Cartesian to spherical and vice versa. In particular, good results can be obtained by changing trigonometric functions used for conversion from different coordinate systems and application of different coordinate system conversion in line with experiments by Paul Nylander in [13].

In figure 2 we present some shapes that we obtained with these experiments. In figure 2c we use applied different trigonometric functions to the inverted Mandelbulb to get $\langle \cos n\phi \sin n\theta, \sin n\phi \tan n\theta, \cos n\theta \rangle$ (compare with the equation 1). In the figure 2d we use further inversion of the coordinates by having $\theta = \operatorname{atan2}\left(\frac{z}{\sqrt{|x^2-y^2|}}\right)$. Similarly, in the figure 2e we used $\phi = (2\sin\frac{z}{r})^{-1}$, the goal was to see how the Mandelbulb changes if the coordinates are converted differently before they are rotated. Finally, in 2f we applied different trigonometric functions as for 2c, but to the original Mandelbulb**.**

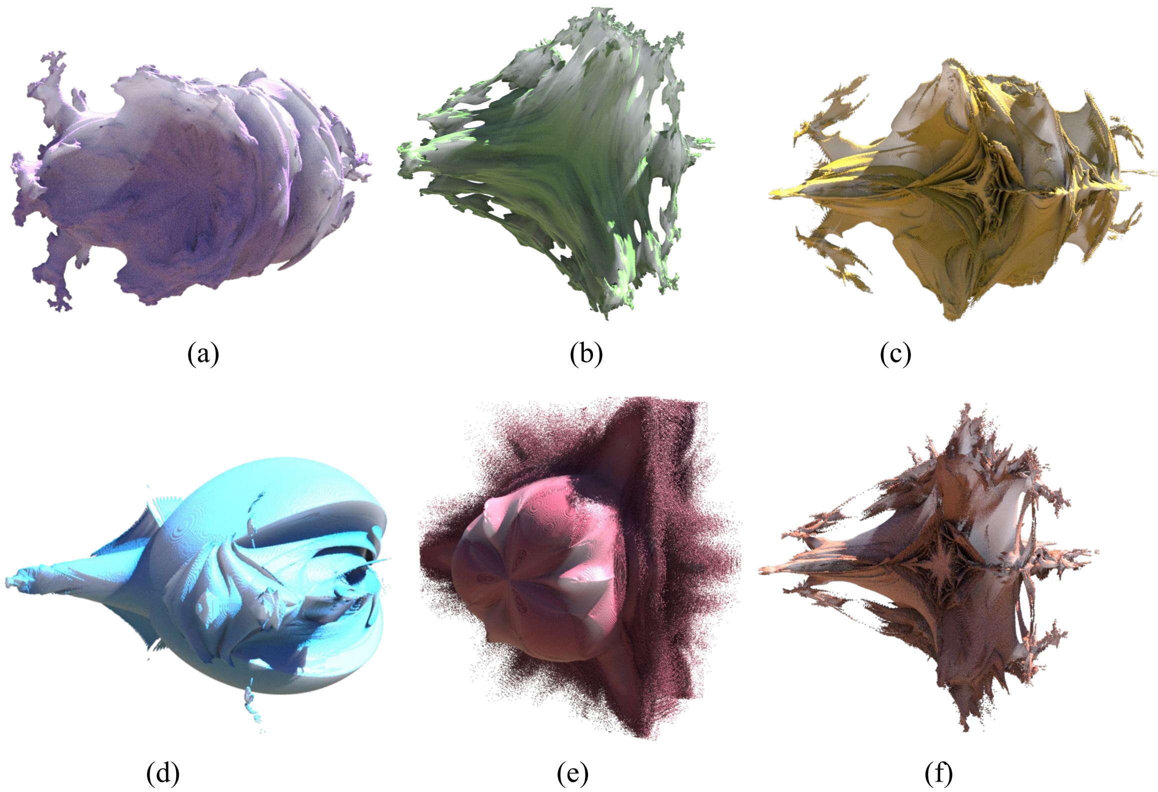


Figure 2: *Hypercomplex fractals of order 2, rendered in Houdini with subsurface scattering: (a) Original Mandelbulb, (b) Inverted Mandelbulb, (c,d) Inverted Mandelbulb with variations, (e) Mandlebulb with adjusted conversion before rotation, (f) Mandelbulb with adjusted conversion after rotation.*

To allow for larger variation in resulting shapes, we added a self-defined kaleidoscopic effect [9]. As noted above, hypercomplex fractals, on the surface, look infinitely complex. However, they are made up of repeating patterns even though they are not easily recognizable to the human eyes. We combined this with easily identifiable patterns, kaleidoscopes, to see the effect it would create. A part of the fractal is taken by using the clip node in Houdini, and mirrored by several sides through a copy node that the user defines. This takes a part of the existing Mandelbulb, mirrors it such that the edges of the parts align perfectly, and then these parts are repeated a given number of times. Due to this method, the kaleidoscopic Mandelbulb can only have an even number of sides, otherwise, one part of the Mandelbulb will not have a mirrored version and thus not connect to the rest of the mesh properly. This creates shapes that are more easily recognizable as symmetrical, which is a good design choice for 3D printing these sculptures as it is easier to balance them. The kaleidoscopic Mandelbulbs can also be edited by simply angling the Mandelbulbs to the user's liking to choose which part of the Mandelbulb will be mirrored. This allows us to create a lot of different kaleidoscopic Mandelbulbs that vary in shape. This created some dramatic changes between the fractals even though they all were produced from the same equation, as seen in Figure 3.

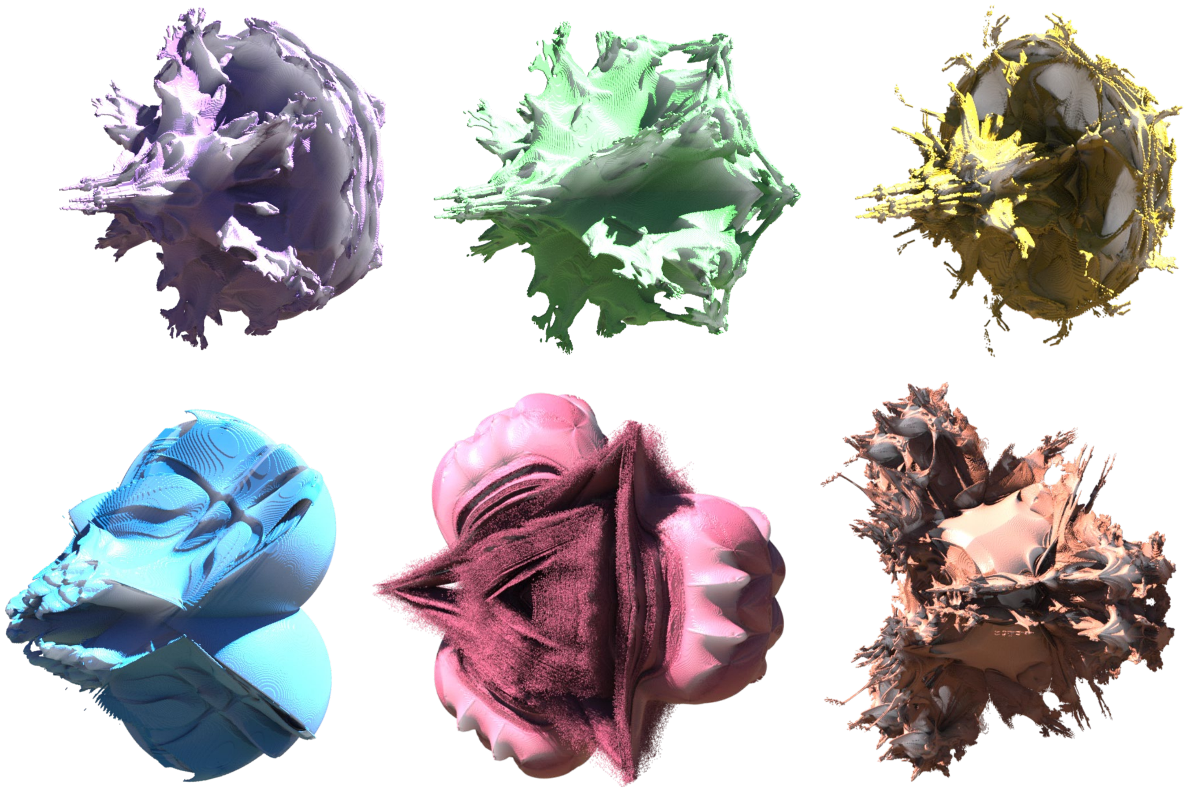


Figure 3: *Mandelbulbs from Figure 2 with the Kaleidoscopic effect applied to them.*

Figure 3 shows the previous Mandelbulbs shown in Figure 2 after applying the kaleidoscopic effect. The Mandelbulbs were purposefully angled in a way where the kaleidoscopic effect creates a visually pleasing shape. With this modification, we can create many hypercomplex fractal shapes by slightly changing the parameters. However, it is important to keep in mind that these shapes are created for 3D printing, therefore there are some limitations in the parameters. It is recommended to keep the power of the Mandelbulbs to a low number as, in this case, the resulting fractal shape has a smoother surface which is more suitable for 3D printing. It is also recommended to keep the number of sides for the kaleidoscopic effect even to avoid visible holes in the geometry and making the resulting object not suitable for 3D printing.

Houdini was used to create these fractals as it allows for quick visualization. The Mandelbulb fractal formula mentioned earlier on by Daniel White and Paul Nylander was converted into VEX code (a Houdini scripting language based on the C language) by referencing Inigo Quilez's algorithm as well [11] using a volume wrangle, which is a low-level node that allows the user to modify voxel values using code. The Vex code takes the Cartesian coordinates (x, y, z) of the points in the volume and converts them into spherical coordinates (r, θ, φ). These points were then scaled and rotated and raised to the user defined power. They were then converted back to Cartesian coordinates and iterated. This node allows the user to edit the power the Mandelbulb is raised to as well as the iteration.

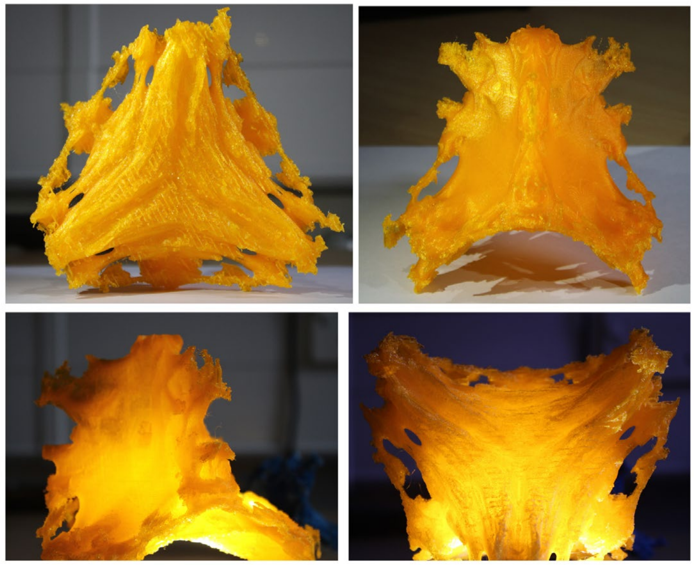


Figure 4: *Inverted 3D Mandelbulb printed in large scale with Ultimaker 2+. The support structure was removed manually.*

Results and analysis

Some of the models presented in the paper were 3D printed. We were using desktop machines of the Ultimaker and Prusa family to fabricate these shapes to demonstrate that the resulting models can be fabricated on a very low budget in a relatively short amount of time.

The original and inverted Mandelbulbs were printed on Ultimaker 3 with a soluble support structure to allow for fine details to be printed and preserved after the support structure was removed. The choice of the sculptures to print was defined by choosing the Mandelbulbs that have a mesh that has no areas that are floating and the visual impact the Mandelbulb has as a 3D printed sculpture. Both sculptures were printed quite small, on average they are 10 centimetres long. The size was chosen to be as such to keep the fine details and at the same time to make the printing process take not too long. These sculptures took about 8 hours to print. One of the objects was also printed in a larger size on single-material Ultimaker 2+ with a support structure that had to be removed manually. Although the single-material printing time on a larger scale was similar to multi-material printing, the post-processing time was annoyingly long, and some details were still lost because the support structure was impossible to remove without affecting the fine details. The resulting sculptures are shown in figures 4, 5 and 6.

The kaleidoscopic Mandlebulb was printed on Prusa MK3S/MMU2S with soluble support. It was printed quite small as well to decrease the duration of printing. Unfortunately, that was proven not the best decision as the resulting print has some very fragile parts, and the model is easier to break.

The availability of the hardware for 3D printing was somehow limited and we could not explore alternative ways to fabricate fractal shapes such as using SLS technology [16] or stereolithography (SLA) using resins [17]. This would be better choice for these sculptures as it is capable of producing high quality prints. This could also allow for translucent materials which would increase the visual aesthetics of the final sculpture and could be used for lighting displays.

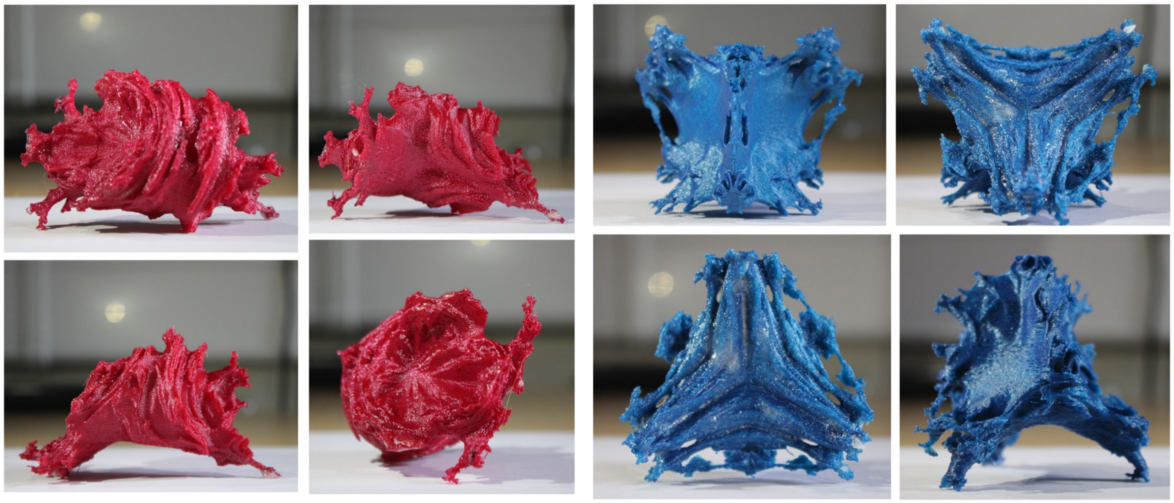


Figure 5: *Mandelbulb and Inverted Mandelbulb 3D printed on Ultimaker 3 with soluble support structure*

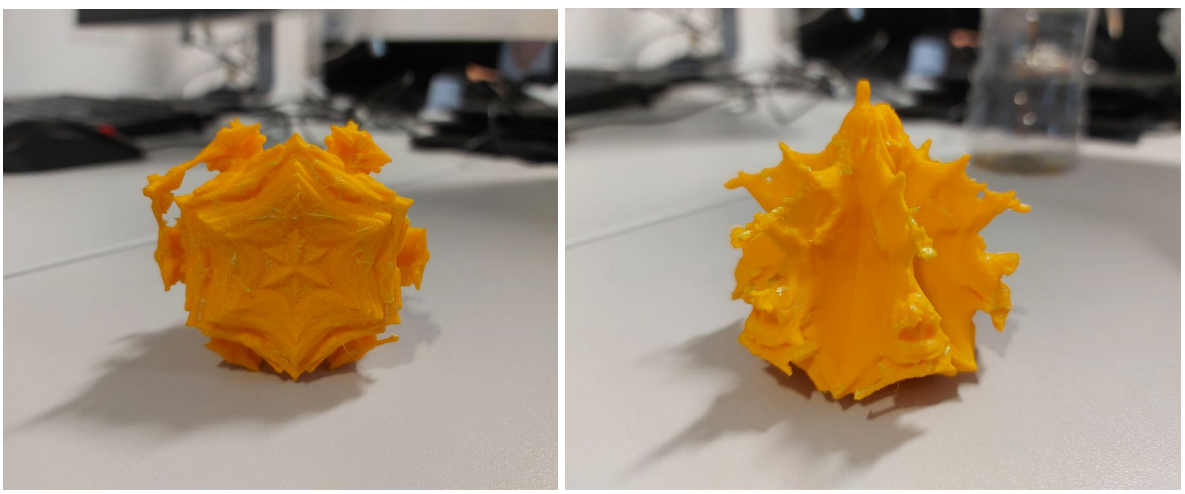


Figure 6: *Kaleidoscopic Mandelbulb printed on Prusa with a layer of PVA between the sculpture and the support.*

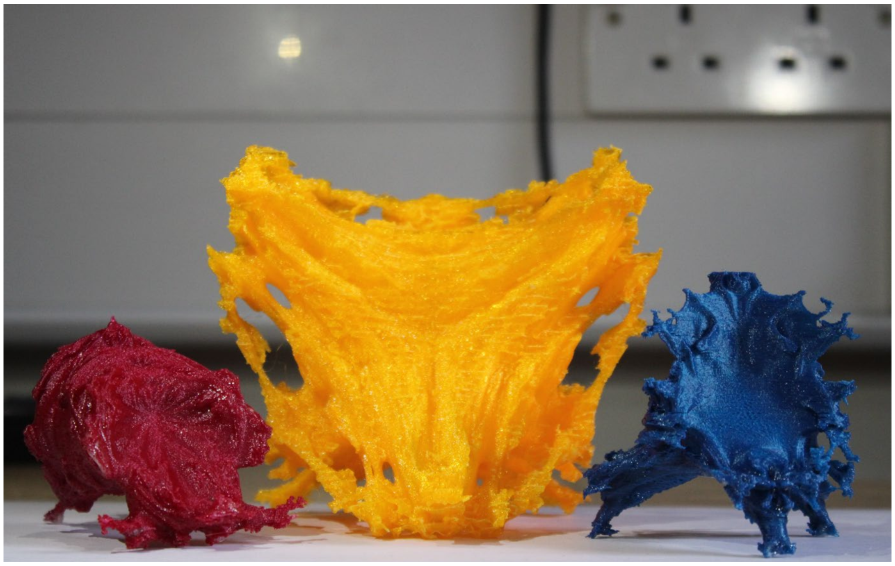


Figure 7: *The final 3D printed Mandelbulbs.*

Conclusions and discussion

Fractal art is an exciting area at the intersection of mathematics and digital art, as it allows to create infinitely complex shapes with a very simple formulation. In this work, we have shown how easy modifications of the Mandelbulb 3D fractal can result in many fractal shapes that can be converted into volume object and subsequently realised as a physical model by using 3D printing. We could achieve that by modifying the formula as well as by application of the kaleidoscopic effect implemented in Houdini software.

While the results are aesthetically pleasing and can serve as the first step into fractal art, we feel that there are some opportunities to develop the idea further and find applications in art. One of the applications of fractals in modern design is fractal jewellery. However, some modifications in shape are required to avoid very thin elements in fractal geometry as well as to ensure the printability of the shape. Automatic detection of non-printable features and correction of the volume object might be a potential future research direction. Another important aspect to consider is the artistic control of fractal shapes. For example, it would be interesting to explore how the artist can make modifications in fractal space and when the shape of the fractal is defined. We can expect that, in a nutshell, it can be done by manual sculpting over the shape of the fractal, but at the same time, an investigation into the semi-automatic methods of doing so can also be a direction for future research. To make the final sculpture more aesthetically pleasing, hand-painting or spray-painting the Mandelbulbs could achieve a more polished look.

References

[1] F. K. Musgrave and B. B. Mandelbrot, "The art of fractal landscapes," in IBM Journal of Research and Development, vol. 35, no. 4, pp. 535-540. 1991.

[2] R.F. Voss (1988). Fractals in nature: From characterization to simulation. In: Peitgen, HO., Saupe, D. (eds) The Science of Fractal Images. Springer, New York, NY.

[3] J. Milner. “Fractal sculpture gallery”. http://www.jimmilnersculpture.co.uk/fractal_stone_sculpture_gallery.htm.

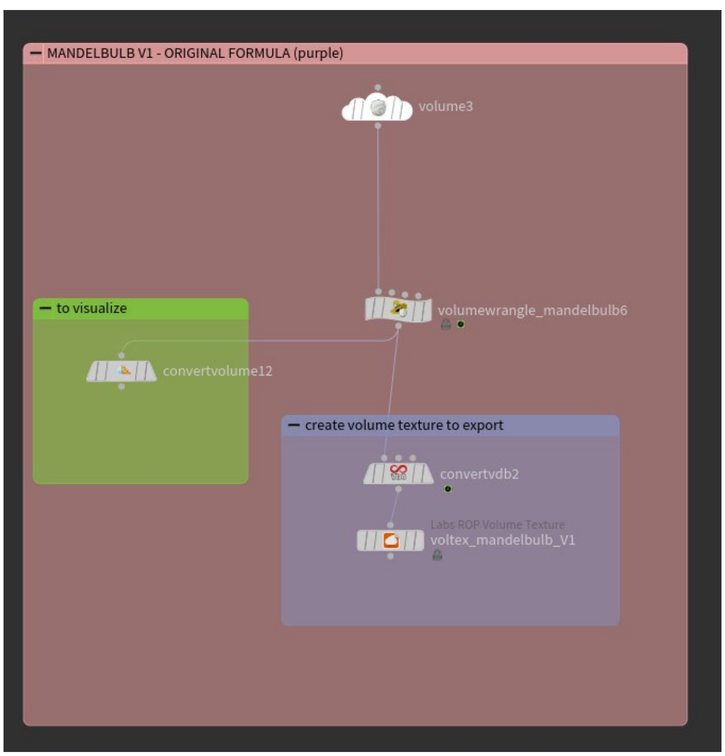
[4] D. Hutchins, O. Riley, J. Erickson, A. Stomakhin, R. Habel, M. Kaschalk. "Big Hero 6: Into the Portal." *ACM SIGGRAPH 2015 Talks*, New York, NY, USA, 2015. doi:10.1145/2775280.2792521

[5] J. Ruis. “Fractal Trigeometry”. https://www.fractal.org/Formula-Mandelbulb.pdf

[6] B.B. Mandelbrot. FRACTAL ASPECTS OF THE ITERATION OF z →Λz(1- z) FOR COMPLEX Λ AND z. Annals of the New York Academy of Sciences, 357: 249-259. 1980
[7] D. White. "The Unravelling of the Real 3D Mandelbrot Fractal". https://www.skytopia.com/project/fractal/2mandelbulb.html
[8] J. Andersson. Surreal Fractal Jewelry and Accessories, https://www.shapeways.com/shops/mandelpieces
[9] I. Peterson. Fragments of Infinity: A Kaleidoscope of Math and Art. Wiley. 2001
[10] I. Quilez "Mandelbulb". https://iquilezles.org/articles/mandelbulb/
[11] American Mathematical Society. "Fractal Art: Beauty and Mathematics". http://www.ams.org/publicoutreach/math-imagery/mandelbrot
[12] B. O'Neal. "Fractal Artist Shows Stunning Strategy in Creating 3D Printe Surreal Chess Set". https://3dprint.com/40665/3d-printed-surreal-chess-set/
[13] P.Nylander. "Hypercomplex fractals." http://www.bugman123.com/Hypercomplex/
[14] R. Brooks, P. Matelski, "The dynamics of 2-generator subgroups of PSL(2,C). Irwin Kra. 1981
[15] A. Sampler, B. Herreta, "The Fractal Pattern of the French Gothic Catherdrals", pp. 251-271. 2014
[16] Xometry. "The Best Cheap 3D Printing Technologies". 2020
[17] Xometry, "3D Technologies Comparison: SLA vs. FDM". 2021
[18] SideFX.

Appendix

Node setup in Houdini:



VEX code for original Mandelbulb formula by Daniel White in Volume Wrangle:

VEXpression

```
vector p = v@P;         //vector positions
float r = 0.0;          // radius
float power = chf("power"); // Power (N)
int maxiter = chi("Max_Iterations");

int num;

for (num = 0; num < maxiter; num++)
{
        //using Inigo Quilez' algorithm but converted to fit my formula

        //converting Daniel White and Paul Nylander's formula for a 3D Mandelbulb
        r = length(p);  // sqrt(p.x*p.x, p.y*p.y, p.z*p.z)
        if (r>2) break;

        //convert to polar coordinates
        float theta = atan2(sqrt(p.x*p.x + p.y*p.y), p.z);
        float phi = atan2(p.y, p.x);


        //scale and rotate the point
        r = pow(r, power);
        theta *= power;
        phi *= power;

        //convert back to cartesian coordinates
        p = r*set(sin(theta)*cos(phi), sin(phi)*sin(theta), cos(theta));
        p += v@P;
}

if(num > 6)
    @density = 100000.0;
```