Minkowski sums of point sets defined by inequalities

A. Pasko [1], O. Okunev [2] & V. Savchenko [1]

[1] *University of Aizu, Japan*

[2] *Universidad Nacional Autónoma de México, México*

Abstract

The existing approaches support Minkowski sums for the boundary, set-theoretic and ray representations of solids. In this paper we consider the Minkowski sum operation in the context of geometric modeling using real functions. The problem is to find a real function $f_3(X)$ for the Minkowski sum of two objects defined by the inequalities $f_1(X) \geq 0$ and $f_2(X) \geq 0$. We represent the Minkowski sum as composition of other operations: the Cartesian product, resulting in a higher dimensional object, and a mapping to the original space. The Cartesian product is realized as an intersection in the higher-dimensional space, using an R-function. The mapping projects the resulting object along n coordinate axes, where n is the dimension of the original space. We discuss the properties of the resulting function and the problems of analytic and numeric implementation, especially for the projection operation. Finally, we apply Minkowski sums to implement offsetting and metamorphosis between set-theoretic solids with curvilinear boundaries.

1 Introduction

This paper deals with the Minkowski sum operation in solid modeling. The Minkowski sum of two geometric objects results from vector sums of all pairs of radius vectors taken from initial objects. It also can be viewed as the union of instances of an object, when placed at all positions corresponding to the points of another object. Minkowski sums are used in solid modeling to generate offsets [1], blends [2], and sweeps [3], to interpolate polyhedral shapes [4] and skeleton-based "implicit" surfaces [5], and to avoid collisions [6, 7]. In a solid modeler, this operation has to result in a valid model, which again can be used as an operand for further geometric transformations and analysis.

We consider the Minkowski sum in the context of geometric modeling using real functions of several variables. The *function representation* (or *F-rep*) defines a geometric object as the set of all points that satisfy the inequality $F(X) \geq 0$ where F is a single real continuous function of several variables. We do not require that the defining function F be polynomial or of any other specific type. The function F may be defined by an analytical expression, or with a function evaluation algorithm, or with scattered data and an appropriate interpolation procedure. This representation combines many different models, such as the classic "implicits", skeleton based "implicits", set-theoretic solids, and volumetric and procedural models [8],[9]. Set-theoretic operations are closed on this representation with the use of R-functions, that is, C^k-continuous functions introduced by Rvachev [10] (see a survey in [11]). Many geometric operations are also closed on F-rep, including blending, offsetting, Cartesian products, sweeping and other (see [8] and [12] for details). These operations generate new real continuous defining functions and provide the closure property of the representation.

The existing approaches support Minkowski sums for the boundary, set-theoretic and ray representations of solids. In this paper we consider the problem of construction of a real continuous function defining the Minkowski sum of two F-rep solids. We reduce the Minkowski sum to simpler operations, the Cartesian product, resulting in a higher-dimensional object, and a mapping to the original space. Then we describe these operations using real functions of several variables. Finally, we discuss the implementation problems, and give some examples. Thus, this paper provides a theoretical solution to the problem. Practical 3D applications of the proposed technique are time consuming and usually require use of parallel or distributed processing.

2 Other works

While Minkowski sums are quite common in image processing, the number of publications on this subject in geometric modeling is rather limited. The main obstacle to the use of this operation are mathematical and computational problems in its implementation for various representations.

Ghosh [3] provides a general framework for Minkowski operations (sums and differences) for boundary-represented 2D and 3D objects. He describes an algorithm for computation of the resulting boundary for sums of two polyhedral objects. A further generalization is done for two planar objects whose boundaries are smooth curves. For the 3D case, the author considers the example of the Minkowski sum of a space curve and a ball. The result is a parametric equation for the swept solid boundary.

In the set-theoretic (or CSG) representation, it is important to provide the point membership classification when introducing a new operation. Parry-Barwick and Bowyer [13] proposed to use for this a multidimensional space. Two operands of the Minkowski sum (a template and a model) span different coordinates in this space. The translational sweep of the template intersects the model considered as a set in the translation dimensions. The Minkowski sum is given by the projection of the intersection into the original space; the projection is computed by a recursive division of the multidimensional space with pruning of the CSG tree that defines the intersection of the sweep and the model. The authors mention that applications of this operation are quite time consuming. In our work, this approach provides a basis for a formal description of the geometric solution and its further functional formulation.

Menon et al. [7] propose to use the definition of the Minkowski sum as the set-theoretic union of instances of one solid translated by radius-vectors of the points of another solid. The approximation of this union is implemented with the ray representation and with a finite number of instances.

An application of Minkowski sums to metamorphosis of skeleton-based implicit surfaces is presented in [5]. The *skeletons* are convex polygonal shapes of various dimensions. The Minkowski sum is applied to the corresponding elements of the skeletons of the operands, resulting in an intermediate skeleton that generates a new implicit surface.

The overview shows that no general technique is available for implementing Minkowski sums of solids defined by arbitrary real functions. In the next section we give the formulation of the problem and describe the proposed solution.

3 Problem statement and proposed solution

For two point sets G_1 and G_2, the Minkowski sum G_3 is defined as follows:

$$G_3 = G_1 + G_2 = \{\, p_3 : \vec{p}_3 = \vec{p}_1 + \vec{p}_2, \quad p_1 \in G_1, \quad p_2 \in G_2 \,\}, \qquad (3.1)$$

where p_1, p_2, p_3 are points, and $\vec{p}_1, \vec{p}_2, \vec{p}_3$ are their radius vectors. This definition depends on the choice of the origin of the radius-vectors; it is easy to see, however, that a change of the origin leads only to a parallel translation of the resulting sum.

Suppose the objects G_1 and G_2 are defined by the inequalities $f_1(X) \geq 0$ and $f_2(X) \geq 0$, where f_1 and f_1 are continuous real functions of a point X. The problem is to find the function f_3 defining the Minkowski sum G_3.

3.1 Geometric formulation

Let us start with the objects in two-dimensional space $\mathbf{R}^2$ for the purposes of exposition. A generalization for higher dimensions is straightforward. We propose a formal description of the geometric solution, which corresponds to the set-theoretic formulation given in [13]. The geometric solution consists of the following steps:

1. Represent the objects G_1 and G_2 in different spaces: G_1 in $\mathbf{R}_1^2$ with coordinates (x_1, y_1), and G_2 in $\mathbf{R}_2^2$ with coordinates (x_2, y_2).

2. The set of all pairs of points of G_1 and G_2 is the Cartesian product $\tilde{G}_3 = G_1 \times G_2$ of G_1 and G_2, which is a subset of the product $\mathbf{R}^4 = \mathbf{R}_1^2 \times \mathbf{R}_2^2$, the Euclidean space with the coordinates (x_1, y_1, x_2, y_2).

3. Let $\mathbf{R}_0^2$ be a two-dimensional Euclidean space with coordinates (x_0, y_0). Define a mapping $T : \mathbf{R}^4 \to \mathbf{R}_0^2$ by the rule: if $X_1 \in \mathbf{R}_1^2$ and

$X_2 \in \mathbf{R}_2^2$, X_1 has the coordinates (x_1, y_1) and X_2 has the coordinates (x_2, y_2), then $T(X_1, X_2)$ is the point in $\mathbf{R}_0^2$ with the coordinates $(x_1 + x_2, y_1 + y_2)$.

4. By the definition of the Minkowski sum, the image G_3 of $\tilde{G}_3$ under the mapping T is the Minkowski sum of G_1 and G_2.

The above procedure can be considered as a geometric formulation of the Minkowski sum operation. We now describe its main steps in terms of real functions.

3.2 Functional formulation

Note that

$$\tilde{G}_3 = G_1 \times G_2 = (G_1 \times \mathbf{R}_2^2) \cap (\mathbf{R}_1^2 \times G_2).$$

Suppose G_1 is defined by a function $f_1(x_1, y_1)$, and G_2 is defined by $f_2(x_2, y_2)$. Then the products $G_1 \times \mathbf{R}_2^2$ and $\mathbf{R}_1^2 \times G_2$ are defined in $\mathbf{R}^4$ by the functions $F_1(x_1, y_1, x_2, y_2)$ and $F_2(x_1, y_1, x_2, y_2)$ on $\mathbf{R}^4$ such that $F_1(x_1, y_1, x_2, y_2) = f_1(x_1, y_1)$ and $F_2(x_1, y_1, x_2, y_2) = f_2(x_2, y_2)$ for all $(x_1, y_1, x_2, y_2) \in \mathbf{R}^4$.

To obtain a function F_3 that defines the intersection $\tilde{G}_3$, we need to apply to the functions F_1 and F_2 an *R-function* for the intersection operation. Thus, the function F_3 that defines $\tilde{G}_3 = G_1 \times G_2$ in $\mathbf{R}^4$ has the form:

$$F_3(x_1, y_1, x_2, y_2) = F_1(x_1, y_1, x_2, y_2) \& F_2(x_1, y_1, x_2, y_2) \tag{3.2}$$

where & stands for an R-function for intersection. Recall that *R-functions* are C^k-continuous real functions defining set-theoretic operations (see [10] and surveys in [8, 11]). The most practically useful R-function for the intersection appears to be

$$F_1 \& F_2 = F_1 + F_2 - \sqrt{F_1^2 + F_2^2} \tag{3.3}$$

Note that this function has C^1-discontinuity only at the points where $F_1 = F_2 = 0$. There are C^k-continuous R-functions for any natural k as well. In Equation 3.2, we define the Cartesian product with an R-intersection of the initial functions as proposed in [8].

Let us now outline how to obtain a function f_3 that defines the set G_3 in $\mathbf{R}_0^2$ from the function F_3 that defines $\tilde{G}_3$ in $\mathbf{R}^4$. We have $G_3 = T(\tilde{G}_3)$, hence a point $X_0 \in \mathbf{R}_0^2$ belongs to G_3 if and only if the preimage $T^{-1}(X_0)$ of X_0 under T meets the set $\tilde{G}_3$. Since a point of $\mathbf{R}^4$ belongs to $\tilde{G}_3$ if and only if the value of F_3 at this point is nonnegative, a point X_0 belongs to G_3 if and only if

$$\max\{\, F_3(X_1, X_2) : T(X_1, X_2) = X_0 \,\} \geq 0,$$

so we can put

$$f_3(X_0) = \max\{\, F_3(X_1, X_2) : T(X_1, X_2) = X_0 \,\}. \tag{3.4}$$

Note that the mapping T is linear; it follows that the preimages of points under T are linear submanifolds in $\mathbf{R}^4$ which are translations of $T^{-1}(0) = \{ (x_1, y_1, x_2, y_2) : x_1 + y_1 = 0, x_2 + y_2 = 0 \}$.

Let $X_0 = T(X_1, X_2)$, $X_0 = (x_0, y_0)$, $X_1 = (x_1, y_1)$ and $X_2 = (x_2, y_2)$. Then by the definition of T, $x_2 = x_0 - x_1$ and $y_2 = y_0 - y_1$; substituting this in Equation 3.2, we get

$$\begin{aligned} &F_3(x_1, y_1, x_0 - x_1, y_0 - y_1) \\ &\qquad = F_1(x_1, y_1, x_0 - x_1, y_0 - y_1) \& F_2(x_1, y_1, x_0 - x_1, y_0 - y_1). \end{aligned} \tag{3.5}$$

Define the function $\bar{F}_3$ on $\bar{\mathbf{R}}^4 = \mathbf{R}_0^2 \times \mathbf{R}_1^2$ by the rule: $\bar{F}_3(x_0, y_0, x_1, y_1) = F_3(x_1, y_1, x_0 - x_1, y_0 - y_1)$ and put $\bar{G}_3 = \{ (x_0, y_0, x_1, y_1) \in \bar{\mathbf{R}}^4 : (x_1, x_0 - x_1, y_1, y_0 - y_1) \in \tilde{G}_3 \}$. It follows from the above argument that $\bar{F}_3$ defines $\bar{G}_3$ in $\bar{\mathbf{R}}^4$; furthermore, G_3 is the projection of $\bar{G}_3$ to the factor $\mathbf{R}_0^2$ of the product $\bar{\mathbf{R}}^4 = \mathbf{R}_0^2 \times \mathbf{R}_1^2$. In this formulation, the Equation 3.4 takes the form

$$f_3(x_0, y_0) = \max\{ \bar{F}_3(x_0, y_0, x_1, y_1) : (x_1, y_1) \in \mathbf{R}_1^2 \}. \tag{3.6}$$

Given (x_0, y_0), we have the following necessary conditions for a point (x_0, y_0, x_1, y_1) where the maximum in the right side is attained:

$$\begin{cases} \dfrac{\partial \bar{F}_3}{\partial x_1}(x_0, y_0, x_1, y_1) = 0 \\ \dfrac{\partial \bar{F}_3}{\partial y_1}(x_0, y_0, x_1, y_1) = 0. \end{cases} \tag{3.7}$$

Solving the last two equations in terms of x_1 and y_1, we can find the required maximum value in the right side of Equation 3.6, and thus derive the required function $f_3(x_0, y_0)$. Of course, the solutions of the equations are generally not unique, and not all of them correspond to a maximum, but in a generic case the number of solutions is finite, so we only have to evaluate $\bar{F}_3$ at these solutions. Note also that we do not really need to find the maximum value of F_3 here; if F_3 is nonnegative at one of these points, that suffices. To establish that a point is not in the projection, however, we need to check that the values of F_3 at all solutions of the system are negative. It should be noted also that checking the values of F_3 at several points appears to be unavoidable, whatever implementation is used; this is suggested by the fact that the projections (and the Minkowski sums) of smooth objects may have singularities, which implies superposition of "maximum" type of several smooth functions.

3.3 Properties of the resulting function

1. The non-compactness of the real line may cause some problems with the application of the above "max-approach" to finding the projection of $\bar{G}_3$. For example, strictly speaking, the maximum in the right side of Equation 3.6 may be never attained for some points $(x_0, y_0) \in \mathbf{R}_0^2$, and we have to use supremum rather than maximum in the definition of f_3 (note

however that this never occurs if (x_0, y_0) lies in G_3 or its sufficiently small neighborhood). Of course, this never happens if instead of considering functions on the whole $\mathbf{R}^2$ we only want to define them on a sufficiently big rectangle containing all figures in question (which is practically always the case). If we still want to have the functions defined on the whole real plane, we need either to take care of the behavior of the functions f_1 and f_2 at the infinity (for example, construct them in such a way that all level lines are compact), or construct f_3 in a sufficiently big rectangle and extend it to a function on the whole plain that is negative outside this rectangle; or, finally, take the maximum over a sufficiently big cell in $\bar{\mathbf{R}}^4$ that contains $\bar{G}_3$ (in this case, in addition to checking the critical points, we may need to consider separately the points of the boundary of this cell).

2. It is easy to deduce from known facts in general topology (see, e.g., 3.12.20 in [14]) that if all level lines of the functions f_1 and f_2 are compact, then the function f_3 is continuous (and in any case, it is continuous in a neighborhood of G_3). It is, however, well-known from the theory of bifurcations that generally, the function f_3 obtained from a smooth function $\bar{F}_3$ as in Equation 3.6 need not be smooth at the points (x_0, y_0) where $\bar{F}_3$ attains the same maximum value at two or more different values of (x_1, y_1). It is not clear under what conditions on f_1 and f_2 we may guarantee that f_3 is smooth; obviously, these conditions must also depend on the choice of the R-function &.

The approach based on 3.6 is practical in the case of a sufficiently simple analytic functional representations, which allow to solve the equations efficiently. Also, an estimate of the error of the calculation of the Minkowski sum in this approach depends on the choice of the functions f_1 and f_2. Some other approaches are presented in the next section.

4 Implementation and examples

In some simple cases it is possible to derive analytically the function that defines the projection. For example, for two unit balls defined by the functions $f_1(x_1, y_1, z_1) = 1 - x_1^2 - y_1^2 - z_1^2$ and $f_2(x_2, y_2, z_2) = 1 - x_2^2 - y_2^2 - z_2^2$, the Minkowski sum is defined, in accordance with Equation 3.5 and Equation 3.3, as follows

$$f_3(x_0, y_0, z_0) = \max\{\, f_1(x_1, y_1, z_1) + f_2(x_0 - x_1, y_0 - y_1, z_0 - z_1) \\ -\sqrt{f_1^2(x_1, y_1, z_1) + f_2^2(x_0 - x_1, y_0 - y_1, z_0 - z_1)} : (x_1, y_1, z_1) \in \mathbf{R}_1^2 \,\}. \tag{4.1}$$

Applying Equation 3.7 extended to three variables (x_1, y_1, z_1), one can derive the following solution:

$$f_3(x_0, y_0, z_0) = 4 - x_0^2 - y_0^2 - z_0^2,$$

which is a correct result, representing the ball of radius 2.

In general case, a numerical projection algorithm is required.

The simplest algorithm is based on using for deciding whether a given point (x_0, y_0) is in the projection of a 4-dimensional body P a uniform two-dimensional grid with the step ε (consisting of the points of the form $(x_0, y_0, x_1 + n\varepsilon, y_1 + m\varepsilon)$ in an appropiate bounding box, and testing the points of the grid for belonging to P; (x_0, y_0) is in the projection if at least one of the points of the grid is in P. To give an estimate of the error in calculation of the Minkowski sum of two figures A and B using this approach, note first that, assuming the absolute precision of the calculation of the functions, we never decide that a point belongs to the sum if in fact it does not, and on the other hand, if $\varepsilon/2$-neighbourhoods of points a and b lie respectively in A and B, then the $\varepsilon/2$-neighbourhood of the point (a, b) is in P, and hence at least one point of the ε-grid meets P, so the point $a + b$ is detected as belonging to the sum of A and B. Denote by A_δ (B_δ), $\delta > 0$ the set of all points a of A (of B) with the property that the δ-neighbourhood of a is in A (respectively, B); we then get the following inclusion for the figure M calculated using the ε-grid algorithm:

$$A_\delta + B_\delta \subset M \subset A + B.$$

where $\delta = \frac{\sqrt{2}}{2}\varepsilon$.

Note that this estimate only depends on A and B, and not on the choice of the representing functions; various modifications of this projection algorithm, such as using quadratic approximations as described below obviously improve the accuracy, but how much exactly cannot be estimated without additional information about the representing functions. Note also that A_δ may be described as the result of removal from A of a δ-neighborhood of the boundary of A. The above estimate remains valid in the case of calculation of the sum of sets of dimension n, with $\delta = \frac{\sqrt{n}}{2}\varepsilon$.

In [15] some algorithms for projection along a one-dimensional subspace are described. The algorithm based on the union of maximal cross-sections has shown the best accuracy and stability. This approximate projection applies set-theoretic union to the interpolation terms between adjacent cross-sections taken with a regular step:

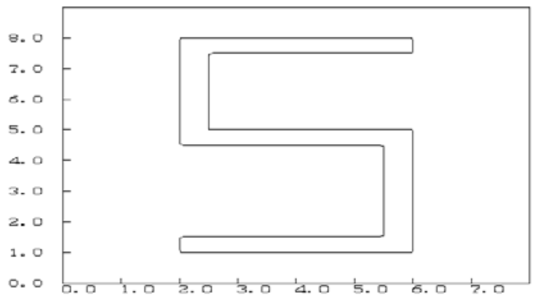
$$f_2(X_{n-1}) = (\ldots(f_{11} \vee f^*_{11}) \vee f^*_{12}) \ldots \vee f^*_{1j}) \ldots \vee f^*_{1,N-2}) \vee f_{1,N}, \quad (4.2)$$

where f_2 defines the projection from E^n onto E^{n-1} along x_i, f_1 defines the initial object, N is the number of cross-sections, $\vee$ stands for an R-function for union, and

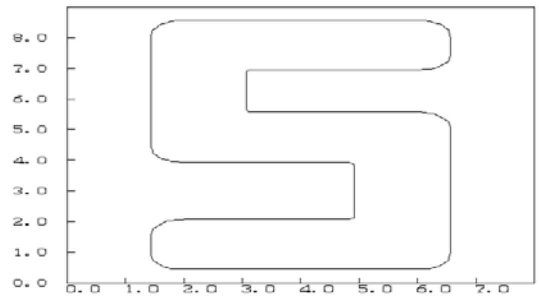
$$f^*_{1j} = f_1(x_1, x_2, \ldots, x_{i-1}, C^*_j, x_{i-1}, \ldots, x_n). \quad (4.3)$$

Here the constant $C^*_j = C_j + C_0 dx_i$ defines the maximum of the function $f_1(X_n)$ between three cross-sections, where C_j is the value of x_i at the j-th grid node with the grid step dx_i. The parameter C_0 is calculated using quadratic interpolation:

$$C_0 = \frac{1}{2} - \frac{f_{1,j+1} - f_{1,j}}{f_{1,j+2} - 2f_{1,j+1} + f_1 j} \quad (4.4)$$



(a) Initial shape



(b) Minkowski sum with a disk, R = 0.6

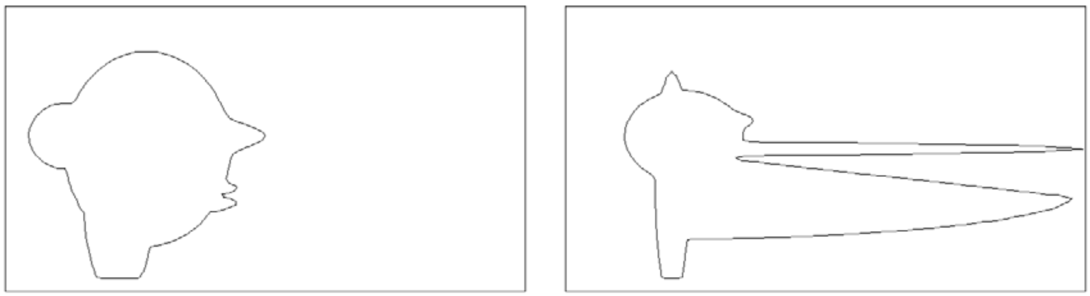
FIGURE 1. Offsetting operation with a Minkowski sum

Note that if $C_0 < 0$, then $f^*_{1j} = f_{1j}$, and if $C_0 > 2$, then $f^*_{1j} = f_{1,j+2}$. This algorithm can be applied to calculation of the Minkowski sums of one-dimensional objects. For the case of 2D objects, we need the projection along a two-dimensional subspace, which reduces to consecutive projections along one-dimensional subspaces. We apply here the union of maximal cross-sections along one-dimensional subspaces that span the two-dimensional subspace, in two nested loops.

This algorithm was used to generate the following examples. Fig. 1 shows a traditional application of the Minkowski sum to generate an offset solid [1]. A constant-radius offset of a 2D, R-functions based set-theoretic solid, is generated by taking the Minkowski sum with a disk. As shown in [4], Minkowski sums can also be effectively applied to define solid-interpolating deformations (or *metamorphosis*). Figs. 2 and 3 illustrate a metamorphosis process based on the following Minkowski sum:

$$f_3(x, y, t) = f_1\left(\frac{x}{1-t}, \frac{y}{1-t}\right) \oplus f_2\left(\frac{x}{t}, \frac{y}{t}\right), \tag{4.5}$$

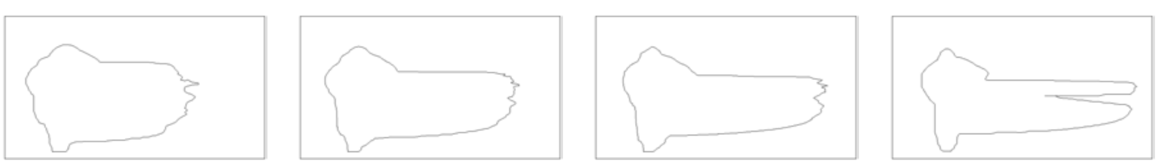
where f_1 and f_2 are the defining functions of the initial and the final shapes, $0 < t < 1$ is the parameter of metamorphosis, and $\oplus$ stands for the functionally defined Minkowski sum proposed above. Fig. 2 shows the initial ($t = 0$) and final ($t = 1$) 2D shapes constructed using set-theoretic and blending operations based on R-functions. Fig. 3 shows the intermediate steps of the metamorphosis defined by Equation 4.5. A survey on shape metamorphosis can be found in [16]. Usually methods of metamorphosis based on the boundary representation are sensitive to the topological differences between two given shapes. Although, function-based models of arbitrary topology can be transformed by a simple linear interpolation between defining functions, there is practically no control of the metamorphosis process. Here, we provided a different approach based on the Minkowski sum. Note that this definition is applicable to F-rep objects of arbitrary topology and dimension, including constructive solids with curvilinear boundaries. The issues of the metamorphosis pro-



(a) t=0: Silly boy (b) t=1: Wise pelican

FIGURE 2. Initial and final shapes for metamorphosis

(a) t=0.1 (b) t=0.2 (c) t=0.3 (d) t=0.4



(e) t=0.5 (f) t=0.7 (g) t=0.8 (h) t=0.9

FIGURE 3. Metamorphosis with Minkowski sums: intermediate shapes

cess control using the time-dependent weighting functions require further investigation.

5 Conclusion

In this paper we consider the Minkowski sum of two point sets defined by continuous real functions. The geometric formulation of the Minkowski sum and the corresponding functional definition are proposed. This allows to apply the Minkowski sum to classic implicits, skeleton-based implicits, and constructive solids defined with set-theoretic, blending and other operations based on R-functions. In particular, this approach helps to solve the quite difficult problem of metamorphosis between two constructive solids with curvilinear boundaries. Minkowski difference and other Minkowski-type operations can be treated in a similar way.

The numerical algorithm applied for the projection is a grid search type algorithm with some additional one-dimensional interpolation. More effective global extremum search algorithms should be considered. Implementation of Minkowski sums for 3D objects will require parallel or distributed processing; we are planning to use networked workstations with the PVM system. Because the numerical procedure is quite time consuming, the procedural definition is not directly applicable in a modern practical modeling system. It seems promising to combine this procedural approach with the voxel-based output of the final Minkowski sum.

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